INTERDIRECTION TESTS FOR REPEATED MEASURES AND ONE-SAMPLE MULTIVARIATE LOCATION PROBLEMS

BY

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To my mother and to

the memory of my father

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INTERDIRECTION TESTS FOR REPEATED MEASURES AND ONE-SAMPLE MULTIVARIATE LOCATION PROBLEMS

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Affine invariant interdirection tests are proposed for a repeated measures problem. The test statistics proposed are applications of the one-sample interdirection sign test and interdirection signed-rank test to a repeated measurement setting. The interdirection sign test has a small sample distribution-free property and includes the two-sided univariate sign test and Blumen's bivariate sign test as special cases. The interdirection signed-rank test includes the two-sided univariate Wilcoxon signed-rank test as a special case. The asymptotic null distributions of the proposed statistics are obtained for the class of elliptically symmetric distributions. In addition, the asymptotic distributions of the proposed statistics under certain contiguous alternatives are obtained for elliptically symmetric distributions with a certain density function form. Comparisons are made between the proposed statistics and several competitors via Pitman asymptotic relative efficiencies and Monte Carlo studies. The interdirection tests proposed appear to be robust. The sign test performs better than the other competitors when the underlying distribution is heavy-tailed or skewed. For normal to light-tailed distributions, the Hotelling's T^2 and signed-rank test have good powers when the variance-covariance structure of the

underlying distribution is non H-type, otherwise ANOVA F and the rank transformation test RT perform better than the others.

An alternative test for the one-sample multivariate location problem is also proposed which extends the univariate signed-rank test to multivariate settings. The test proposed is somewhat like applying the interdirection sign test to the sums of pairs of observed vectors. It includes the two-sided univariate Wilcoxon signed-rank test as a special case. The asymptotic distributions of the proposed statistic under the null hypothesis and under certain contiguous alternatives are obtained for a class of elliptically symmetric distributions. Comparisons are made between the proposed statistic and Hotelling's T^2 via Pitman asymptotic relative efficiencies. The signed sum test proposed performs better than Hotelling's T^2 when the underlying distribution is heavy-tailed. However, for normal to light-tailed distributions, the Hotelling's T^2 performs slightly better than the proposed test.

CHAPTER 1
INTRODUCTION

In this dissertation we investigate test statistics for certain repeated-measures and one-sample multivariate location problems. For the repeated-measures problem, we let $\underline{Y}_1, \ldots, \underline{Y}_n$ be independently and identically distributed as $\underline{Y}$, where $\underline{Y} = (Y_1, \ldots, Y_p)^T$ is from a p-dimensional, $p \geq 2$, absolutely continuous population. For each subject i, we shall regard $\underline{Y}_i$ as repeated measures with one observation for each of the p treatments. We use the general mixed model with a subject by treatment interaction. In vector form, we consider the model (see, e.g., Winer [1971], p. 278),

$$\underline{Y}_i = \beta_i \underline{1}_p + \underline{\tau} + \underline{\beta\tau}_i + \underline{\varepsilon}_i, \ i = 1, \ldots, n, \tag{1.1.1}$$

where $\underline{1}_p$ is the p×1 vector of 1's, $\underline{\tau} = (\tau_1, \ldots, \tau_p)^T$ is the vector of fixed treatment effects, and random variables β_i, $\underline{\beta\tau}_i$, $\underline{\varepsilon}_i$ for i = 1, …, n are all mutually independent. (More details about this model will be given in section 2.1.) Note that the variance-covariance matrix of $\underline{\beta\tau}_i$ is probably not H-type, to be described later. We are concerned with the test of equal treatment effects, which in model (2.1.1) can be described as:

$$H_0: \tau_1 = \tau_2 = \ldots = \tau_p \quad \text{versus} \quad H_a: \tau_j \neq \tau_{j'} \text{ for some } j \neq j'. \tag{1.1.2}$$

We first consider several parametric statistics for this problem when the population is p-variate normal. Probably the most well-known parametric procedure for this problem is based on Hotelling (1931) T^2 test. Define the (p-1)-variate random vector $\underline{Z}_i$ by

$$\underline{Z}_i = (Z_{i1}, \ldots, Z_{i\,p-1})^T = (Y_{i1} - Y_{ip}, \ldots, Y_{i\,p-1} - Y_{ip})^T, \ i = 1, \ldots, n.$$

The test of (1.1.2) can be carried out by the Hotelling's T^2 statistic computed from the mean vector and sample variance-covariance matrix of the vectors of differences $\underline{Z}_i$'s. This is due to Hsu (1938). The Hotelling's T^2 is then defined as

$$T^2 = T^2(\underline{Z}_1, \ldots, \underline{Z}_n) = n\,\overline{\underline{Z}}^T (\hat{\underline{\Sigma}})^{-1} \overline{\underline{Z}}, \tag{1.1.3}$$

where

$$\overline{\underline{Z}} = \frac{1}{n} \sum_{i=1}^{n} \underline{Z}_i, \text{ and } \hat{\underline{\Sigma}} = \frac{1}{n-1} \sum_{i=1}^{n} (\underline{Z}_i - \overline{\underline{Z}})(\underline{Z}_i - \overline{\underline{Z}})^T.$$

Under H_0, T^2 is asymptotically Chi-square with p-1 degrees of freedom. If the underlying population is p-variate normal, then the null distribution of T^2 is a multiple of the F-distribution with p-1 numerator degrees of freedom and n-p+1 denominator degrees of freedom. Hotelling's T^2 is invariant with respect to nonsingular linear transformations of the observations $\underline{Z}_i$, $i = 1, \ldots, n$. That is, if $\underline{D}$ is any nonsingular $(p-1)\times(p-1)$ matrix, then

$$T^2(\underline{D}\,\underline{Z}_1, \ldots, \underline{D}\,\underline{Z}_n) = T^2(\underline{Z}_1, \ldots, \underline{Z}_n). \tag{1.1.4}$$

We shall call this invariance property affine-invariance. This appealing invariance property ensures that the value of the test statistic remains unchanged following rotations of the observations about the origin, reflections of the observations about a (p-2)-dimensional hyperplane through the origin, or changes in scale. Hence the performance of Hotelling's T^2 test will not depend on the structure of the population variance-covariance matrix or the direction of shift.

Another parametric procedure for this problem is the classic ANOVA F test. The test is based on the original observations Y_{ij}, $1 \le i \le n$, $1 \le j \le p$, and is defined as

$$F = \frac{n \sum_{j=1}^{p} (\overline{Y}_{.j} - \overline{Y}_{..})^2}{\frac{1}{n-1} \sum_{i=1}^{n} \sum_{j=1}^{p} (Y_{ij} - \overline{Y}_{i.} - \overline{Y}_{.j} + \overline{Y}_{..})^2}, \tag{1.1.5}$$

where $\overline{Y}_{.j} = \frac{1}{n}\sum_{i=1}^{n} Y_{ij}$, $\overline{Y}_{..} = \frac{1}{p}\sum_{j=1}^{p} \overline{Y}_{.j}$, and $\overline{Y}_{i.} = \frac{1}{p}\sum_{j=1}^{p} Y_{ij}$. If the underlying population is p-variate normal with variance-covariance matrix of the form

$$\underline{\Sigma} = \sigma^2 \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & \ddots & \rho & \vdots \\ \vdots & \rho & \ddots & \rho \\ \rho & \cdots & \rho & 1 \end{bmatrix}, \tag{1.1.6}$$

then the null distribution of F is F-distribution with p-1 numerator degrees of freedom and (n-1)(p-1) denominator degrees of freedom. This test for (1.1.2) under the variance-covariance matrix (1.1.6) was obtained by Wilks (1946) from the generalized likelihood ratio principle when the underlying population is p-variate normal. The matrix in (1.1.6) is said to have compound symmetry. While compound symmetry is a sufficient condition for the test statistic F to have an exact F-distribution, it is not a necessary one. Huynh and Feldt (1970) have found a necessary and sufficient condition, which may be expressed in three alternative forms (see Morrison [1976], p.152):

(1) The population variance-covariance matrix $\underline{\Sigma} = (\sigma_{jj'})$, $1 \le j, j' \le p$, has the pattern defined by

$$\sigma_{jj'} = \begin{cases} \alpha_j + \alpha_{j'} + \lambda & \text{if } j = j' \\ \alpha_j + \alpha_{j'} & \text{if } j \ne j' \end{cases}, \tag{1.1.7}$$

where $\lambda, \alpha_1, \ldots, \alpha_p$ are (p+1) arbitrary constants such that the resultant matrix is positive definite.

(2) All possible differences $Y_j - Y_{j'}$ $(j \ne j')$ of the response variates have the same variance.

(3) Define ε, a function of the elements of $\underline{\Sigma}$, by

$$\varepsilon = \frac{p^2(\bar{\sigma}_d - \bar{\sigma}_{..})^2}{(p-1)\left(\sum_{j=1}^{p}\sum_{j'=1}^{p}\sigma_{jj'}^2 - 2p\sum_{j=1}^{p}\bar{\sigma}_{j.}^2 + p^2\bar{\sigma}_{..}^2\right)}, \tag{1.1.8}$$

where $\bar{\sigma}_d = \frac{1}{p}\sum_{j=1}^{p}\bar{\sigma}_{jj}$, $\bar{\sigma}_{j.} = \frac{1}{p}\sum_{j'=1}^{p}\bar{\sigma}_{jj'}$, and $\bar{\sigma}_{..} = \frac{1}{p}\sum_{j=1}^{p}\bar{\sigma}_{j.}$.

Then an alternative statement of the necessary and sufficient condition is that $\varepsilon = 1$.

It may be noted that the scalar factor ε is the multiplicative adjustment factor for the degrees of freedom proposed by Box (1954) and by Greenhouse and Geisser (1959). Any matrix whose elements satisfy (1.1.7) is referred to as a matrix of Type H. If the underlying population is p-variate normal with H-type variance-covariance matrix, the power of the F test will always be greater than that of the Hotelling's T^2 test for the same alternative because the second degrees-of-freedom parameter of the F test is greater than that of Hotelling's T^2 test. However, unlike Hotelling's T^2, ANOVA F does not have affine-invariance. Thus, the performance of F test depends strongly on the structure of the variance-covariance matrix of the underlying population.

Morrison (1972) proposed several tests for this problem under various assumptions on the variance-covariance matrix. The tests include the usual Hotelling-Hsu statistic and the classical ANOVA F test as special cases.

Many nonparametric competitors to Hotelling's T^2 have been proposed. A well-known statistic is a rank test due to Friedman (1937). His test statistic is based entirely on within-block information and ignores between-block information. Let R_{ij} denote the rank assigned to Y_{ij} within subject i. Let R_j represent the sum of the ranks associated with treatment j, i.e., $R_j = \sum_{i=1}^{n} R_{ij}$, j = 1, ..., p. The Friedman test statistic for this case with no ties is

$$S = \frac{12}{np(p+1)}\sum_{j=1}^{p} R_j^2 - 3n(p+1). \tag{1.1.9}$$

Under H_0, the test S is asymptotically Chi-square with p-1 degrees of freedom. (See, e.g., Hollander and Wolfe [1973], p.140.) Define

$$F_S = \frac{(n-1)S}{n(p-1)-S}\,.$$

Then under H_0, the test F_S is compared with an F-distribution with degrees of freeedom p-1 and (n-1)(p-1). A more accurate approximation to Friedman's test is proposed by Jensen (1977). Iman and Davenport (1980) proposed two new approximations and also pointed out that the F approximation is better than the χ^2 approximation.

Another rank test was proposed by Koch (1969). His test statistic used the ranks of the aligned observations, obtained by subtracting from each observation the average of the observations for that block. This alignment process will eliminate or at least reduce the block effect. To introduce his test, let us define

$$R_{ij} = \text{rank of } (Y_{ij}-\bar{Y}_{i.}) \text{ among } Y_{11}-\bar{Y}_{1.}, \ldots, Y_{np}-\bar{Y}_{n.},$$

$$\bar{R}_{.j} = \frac{1}{n}\sum_{i=1}^{n} R_{ij} \text{ and } \bar{R}_{i.} = \frac{1}{p}\sum_{j=1}^{p} R_{ij}.$$

Then the test statistic has the form

$$\tilde{W}^* = \frac{n\sum_{j=1}^{p}(\bar{R}_{.j} - \frac{np+1}{2})^2}{\frac{1}{n(p-1)}\sum_{i=1}^{n}\sum_{j=1}^{p}(R_{ij} - \bar{R}_{i.})^2}\,. \tag{1.1.10}$$

If $\underline{Y}_1, \ldots, \underline{Y}_n$ are i.i.d., then under the null hypothesis of the exchangeability of the components of $\underline{Y}_i = (Y_{i1}, \ldots, Y_{ip})^T$, the test $\tilde{W}^*$ has an asymptotic Chi-square distribution with degrees of freedom p-1.

Unlike the Friedman test, which depends entirely on within-block rankings, Quade (1979) considered a p-sample extension of the Wilcoxon signed rank test by taking

advantage of the between-block information. This is done by considering weights assigned to each block on the basis of some measure of within-block sample variation, such as the range, standard deviation, mean deviation, or interquartile difference. To illustrate this test statistic, let us define

$D_i = D(\underline{Y}_i)$, a location-free statistic that measures the variability within the ith block,

Q_i = rank of D_i among $D_1, \ldots, D_n$, and

R_{ij} = rank of Y_{ij} among $Y_{i1}, \ldots, Y_{ip}$.

Quade's procedure, based on weighted within-block ranks, is defined as

$$W = \frac{72\sum_{j=1}^{p}\left[\sum_{i=1}^{n} Q_i R_{ij}\right]^2}{p(p+1)n(n+1)(2n+1)} - \frac{9(p+1)n(n+1)}{2(2n+1)}$$

$$= \frac{72\sum_{j=1}^{p}\left[\sum_{i=1}^{n} Q_i (R_{ij} - \frac{k+1}{2})\right]^2}{p(p+1)n(n+1)(2n+1)}. \tag{1.1.11}$$

Under the null hypothesis of the exchangeability of the components of $\underline{Y}_i$, the test W has an asymptotic Chi-square distribution with p-1 degrees of freedom.

A natural nonparametric analog of ANOVA F test was proposed by Iman, Hora and Conover (1984). Their procedure is first to transform all the observations to ranks from 1 to np and then apply the parametric ANOVA F test to the ranks. This approach retains both the within- and between-block information. Defining

R_{ij} = rank of Y_{ij} among $Y_{11}, \ldots, Y_{np}$,

the rank transformation test is defined as

$$RT = \frac{n \sum_{j=1}^{p} (\bar{R}_{.j} - \bar{R}_{..})^2}{\frac{1}{n-1} \sum_{i=1}^{n} \sum_{j=1}^{p} (R_{ij} - \bar{R}_{i.} - \bar{R}_{.j} + \bar{R}_{..})^2}, \tag{1.1.12}$$

where $\bar{R}_{.j} = \frac{1}{n} \sum_{i=1}^{n} R_{ij}$, $\bar{R}_{i.} = \frac{1}{p} \sum_{j=1}^{p} R_{ij}$ and $\bar{R}_{..} = \frac{1}{p} \sum_{j=1}^{p} R_{.j}$. Assuming Y_{ij}'s are mutually independent, and considering

$$H_0 : F_{ij} = F_i \text{ for } i = 1, \ldots, n \text{ and } j = 1, \ldots, p,$$

where F_{ij} is the distribution function of Y_{ij}, then the asymptotic null distribution of (p-1)RT is Chi-square with p-1 degrees of freedom under suitable conditions. Their simulations showed that the behavior of test RT is closely approximated by the F-distribution with (p-1) and (n-1)(p-1) degrees of freedom. Comparisons made among ANOVA F test, Friedman's test (S), Quade's test (W), and rank transformation test (RT) via Monte Carlo studies showed that the F test had the most power for normal distributions, the Quade's test and F test were almost equivalent and gave the best results for uniform distribution, the Friedman's test and the RT test gave similar results and were best for the Cauchy distribution, and the RT test has the most power for double exponential and lognormal distributions. Hora and Iman (1988) developed the limiting noncentrality parameters of the rank transformation statistic and some other tests, which were then evaluated to make comparisons among those tests via Pitman asymptotic relative efficiencies.

Agresti and Pendergast (1986) also considered a test that is appropriate when the null hypothesis (1.1.2) is expressed as the exchangeability of the components of $\underline{Y}_i$. Their procedure utilizes a single ranking of the entire sample. Let

$$R_{ij} = \text{rank of } Y_{ij} \text{ among } Y_{11}, \ldots, Y_{np}, \bar{R}_{.j} = \frac{1}{n} \sum_{i=1}^{n} R_{ij},$$

$$\rho = \mathrm{Corr}(R_{ij}, R_{ij'}),\ j \neq j', \text{ and } \sigma^2 = \mathrm{Var}(R_{ij}).$$

The test statistic is based on

$$T = \frac{n \sum_{j=1}^{p} (\bar{R}_{.j} - \frac{np+1}{2})^2}{\sigma^2(1-\rho)} . \tag{1.1.13}$$

This test includes Koch's test and the rank transformation test as special cases. They argued that under H_0 their statistic has an asymptotic Chi-square distribution with p-1 degrees of freedom if the asymptotic distribution of $\underline{\bar{R}} = (\bar{R}_{.1}, \ldots, \bar{R}_{.p-1})^T$ is (p-1)-variate normal. Since they did not present conditions guaranteeing this normality, Kepner and Robinson (1988) concluded the work of Agresti and Pendergast by determining reasonably sufficient conditions for $\underline{\bar{R}}$ to have a (p-1)-variate normal limiting distribution.

In analogy to the Iman, Hora and Conover (1984) proposal of a rank transformed version of ANOVA F test, Agresti and Pendergast (1986) considered a rank transformed version of Hotelling's test. This procedure is appropriate when the hypothesis of no treatment effects is more broadly expressed as the marginal homogeneity condition $F_1 = F_2 = \ldots = F_p$, where $F_1, \ldots, F_p$ denote the one-dimensional marginal distribution of $\underline{Y} = (Y_1, \ldots, Y_p)^T$. Their statistic is based on

$$RT_H = n\, \underline{\bar{R}}^T \underline{S}^{-1} \underline{\bar{R}}, \tag{1.1.14}$$

where

$$\underline{\bar{R}} = \frac{1}{n} \sum_{i=1}^{n} \underline{R}_i,\ \ \underline{R}_i = (R_{i1}-R_{ip}, \ldots, R_{i\,p-1}-Rip)^T, \text{ and } \underline{S} = \frac{1}{n-1} \sum_{i=1}^{n} (\underline{R}_i - \underline{\bar{R}})(\underline{R}_i) - \underline{\bar{R}})^T.$$

Their simulations showed treating the null distribution of RT_H as a multiple of F-distribution with p-1 numerator degrees of freedom and n-p+1 denominator degrees of freedom is a reasonable approximation. They also argued that under H_0, their statistic has

an asymptotically Chi-square distribution with p-1 degrees of freedom if the asymptotic distribution of $(\bar{R}_{.1}, \ldots, \bar{R}_{.p-1})^T$ is (p-1)-variate normal. Their simulations showed that the RT and RT_H statistics behaved much like their parametric analogs.

We now consider the multivariate tests for the one-sample location problem. As a first step, we let $\underline{X}_1, \ldots, \underline{X}_n$ be i.i.d. as $\underline{X} = (X_1, \ldots, X_p)^T$, where $\underline{X}$ is from a p-variate absolutely continuous population with location parameter $\underline{\theta}^*$ (p×1). We would like to test

$$H_0 : \underline{\theta}^* = \underline{0} \quad \text{versus} \quad H_a: \underline{\theta}^* \neq \underline{0}. \tag{1.1.15}$$

Here $\underline{0}$ is used without loss of generality, since $H_0 : \underline{\theta}^* = \underline{\theta}_0$ can be tested by subtracting $\underline{\theta}_0$ from each observation $\underline{X}_i$ and testing whether these differences $(\underline{X}_i - \underline{\theta}_0)$'s are located at $\underline{0}$.

The classical procedure used in this setting is Hotelling's T^2, which is defined as

$$T^2 = T^2(\underline{X}_1, \ldots, \underline{X}_n) = n\,\bar{\underline{X}}^T(\hat{\underline{\Sigma}})^{-1}\bar{\underline{X}},$$

where

$$\bar{\underline{X}} = \frac{1}{n}\sum_{i=1}^{n}\underline{X}_i, \text{ and } \hat{\underline{\Sigma}} = \frac{1}{n-1}\sum_{i=1}^{n}(\underline{X}_i-\bar{\underline{X}})(\underline{X}_i-\bar{\underline{X}})^T.$$

If the underlying population is p-variate normal, then the null distribution of T^2 is a multiple of the F-distribution with p numerator degrees of freedom and n-p denominator degrees of freedom. The affine-invariant property of Hotelling's T^2 is discussed earlier on page 2.

Many nonparametric procedures have been proposed. The most popular statistic is the component sign test, which is a nonparametric analog of Hotelling's T^2 using signs of X_{ij}'s, $1 \le i \le n$, $1 \le j \le p$. (See, e.g., Randles [1989], p. 1045.) Let

$$S_j = \sum_{i=1}^{n} \operatorname{sgn}(X_{ij}), \text{ and } \underline{S} = (S_1, \ldots, S_p)^T,$$

where $sgn(x) = 1(0, -1)$ for $x >(=, <) 0$. The test is based on

$$S_n^* = \underline{S}^T (n\hat{\underline{W}})^{-1} \underline{S}, \tag{1.1.16}$$

where $\hat{\underline{W}} = (w_{jj'})$, $w_{jj'} = \frac{1}{n} \sum_{i=1}^{n} sgn(X_{ij}) sgn(X_{ij'})$, for $1 \le j, j' \le p$.

Under H_0, the test statistic S_n^* is asymptotically Chi-square with p degrees of freedom. Sen and Puri (1967) considered other score function versions of this statistic as well. Bennett (1962) considered a similar sign test for the problem of testing the equality of location parameters in two p-variate ($p \le 4$) populations. Chatterjee (1966) also studied a similar procedure for a bivariate case. In the paper of Bickel (1965), he proposed several general test statistics based on Hotelling's T^2. They are Hotelling's tests of type I, M^2 and W^2, and Hotelling's tests of type II, $\hat{M}^2$ and $\hat{W}^2$. His p-variate test statistics M^2 and $\hat{M}^2$, and W^2 and $\hat{W}^2$ are quadratic forms involving coordinate-wise sign and Wilcoxon signed rank statistics, respectively.

Unlike Hotelling's T^2, the above competitors to T^2 are not invariant under nonsingular linear transformations of the observations. Consequently, their power and efficiency depend on the direction of shift and the variance-covariance matrix of the alternative distribution. In an effort to overcome this problem, Dietz (1982) proposed sign and signed rank tests for bivariate location settings. Her tests are three-step procedures. First, a certain transformation is applied to the observations. Second, new coordinate axes are chosen. Third, standard sign and signed rank tests are performed using the transformed and rotated observations. In Dietz (1984), she extended those results to general linear signed rank tests for multivariate location problems. Here, however, a two-stage procedure is recommended, corresponding to steps one and three of the bivariate case. The resulting test statistics are no longer invariant under linear transformations, but for elliptically symmetric alternatives, their asymptotic efficiencies are independent of the direction of shift and the variance-covariance matrix. Her simulation study showed that linear

transformations have little effect on the small sample power of the tests for nearly degenerate distributions.

Two well-known bivariate sign tests are due to Hodges (1955) and Blumen (1958). Their procedures are affine-invariant and have a distribution-free property. Joffe and Klotz (1962) presented an expression for the exact null distribution of the Hodges bivariate sign test. They also computed the Bahadur limiting efficiency of the test relative to the Hotelling's T^2 test for normal alternatives. Killeen and Hettmansperger (1972) made an exact Bahadur efficiency comparison of Hotelling's T^2 with respect to both Hodges' and Blumen's bivariate sign tests. Klotz (1964) obtained exact power for the bivariate sign tests of Hodges and Blumen under normal alternatives and therefore permitted comparisons of the two tests for sample sizes n = 8 through 12.

The procedure proposed by Bennett (1964) for the bivariate case is a signed-rank test generalizing Wilcoxon's univariate signed-rank test. This test is not affine-invariant.

Another affine-invariant bivariate rank test was introduced by Brown and Hettmansperger (1985). Their statistic is based on the gradient of Oja's measure of scatter (Oja, 1983). Letting $A(\underline{X}_i, \underline{X}_j; \underline{\theta}^*)$ denote the area of the triangle formed with $\underline{X}_i$, $\underline{X}_j$ and $\underline{\theta}^*$ as vertices, define

$$T(\underline{\theta}^*) = \sum_{i<j}\sum A(\underline{X}_i, \underline{X}_j; \underline{\theta}^*). \tag{1.1.17}$$

This is the Oja measure of scatter. The value $\hat{\underline{\theta}}^*$ which minimizes $T(\underline{\theta}^*)$ is the Oja generalized median of the bivariate sample. Brown and Hettmansperger proposed the use of $\underline{Q}_n(\underline{\theta}^*) = \partial T(\underline{\theta}^*)$, the vector of partial derivatives of $T(\underline{\theta}^*)$. The generalized median of Oja is the value $\hat{\underline{\theta}}^*$ which minimizes $|\underline{Q}_n(\underline{\theta}^*)|$. Define

$$\underline{S} = \sum_{i=1}^{n} \underline{Q}_{2n}(-\underline{X}_i), \text{ and } \underline{C} = \sum_{i=1}^{n} \underline{Q}_{2n}(-\underline{X}_i)\underline{Q}_{2n}^T(\underline{X}_i), \tag{1.1.18}$$

where $Q_{2n}(-\underline{X}_i)$ is computed using observations $\underline{X}_1, \ldots, \underline{X}_n$, and their projections through the origin, $-\underline{X}_1, \ldots, -\underline{X}_n$. Their test for (1.1.15) is defined as $\underline{S}^T\underline{C}^{-1}\underline{S}$. Under H_0, the test statistic is asymptotically Chi-square with 2 degrees of freedom.

Oja and Nyblom (1989) also studied the bivariate location problem. Their tests are analogs of the univariate sign test. Denote the direction angle of $\underline{X}_i$ by θ_i. Then $\theta_i^* = \theta_i+\pi$ (mod 2π) is the direction angle of $-\underline{X}_i$. Write $\theta_1' < \theta_2' < \ldots < \theta_{2n}'$ for the ordered angles in the set $\{\theta_1, \ldots, \theta_n, \theta_1^*, \ldots, \theta_n^*\}$. Define

$$
\begin{aligned}
Z_i = \; & 1 \quad \text{if } \theta_i' \in \{\theta_1, \ldots, \theta_n\} \\
& 0 \quad \text{if } \theta_i' \in \{\theta_1^*, \ldots, \theta_n^*\},\ i = 1, \ldots, n,
\end{aligned}
\tag{1.1.19}
$$

and

$$Z_{n+i} = 1-Z_i\ ,\ i = 1, 2, \ldots. \tag{1.1.20}$$

The vector $\underline{Z} = (Z_1, \ldots, Z_n)^T$ indicates which of the observations lie above or below the horizontal axis. They proposed using test statistics based on $\underline{Z}$. The test statistics are distribution-free and affine-invariant, and include Hodges (1955) and Blumen (1958) sign tests as special cases. Also, they proposed some new intuitively appealing tests. A general class of these invariant sign test statistics is

$$\sum_{k=0}^{n-1} \Big[\sum_{i=1}^{n} (Z_{k+i} - \tfrac{1}{2})\cdot h(i/n)\Big]^2 \tag{1.1.21}$$

where h is a suitably chosen score function.

In the recent study Randles (1989) proposed an interdirection sign test for this problem. His test statistic included the two-sided univariate sign test and the Blumen (1958) bivariate sign test as special cases. Also, Peters and Randles (1990) suggested a signed-rank test based on interdirections, which includes the two-sided Wilcoxon signed-

rank test as a special case. The bivariate case of their statistic was considered in the dissertation of Peters (1988). The interdirection sign test and the interdirection signed-rank test will be described in detail in Chapters 2 and 3, respectively, where they are applied to a repeated measures problem.

In this dissertation, the interdirection sign test for a repeated measures problem is defined in Chapter 2. The asymptotic distributions of the test under H_0 and under certain contiguous alternatives are obtained in sections 2.2 and 2.3, respectively. The Pitman asymptotic relative efficiencies of the test relative to Hotelling's T^2 are presented in the last section. In Chapter 3, the interdirection signed-rank test for the same problem is described, with its asymptotic distributions obtained in section 3.2 and the evaluations of the ARE of the signed-rank test relative to Hotelling's T^2 established in the section 3.3. Comparisons of several competing procedures are made in Chapter 4 via Monte Carlo studies. An alternative test for the one-sample multivariate location problem is proposed in Chapter 5. Some useful intermediate results are presented in section 5.2. The asymptotic distributions of the test under H_0 and under certain contiguous alternatives are developed in sections 5.3 and 5.4, respectively. Finally, in section 5.5, we evaluate the ARE of the proposed test relative to Hotelling's T^2.

CHAPTER 2
A MULTIVARIATE SIGN TEST BASED ON INTERDIRECTIONS FOR REPEATED-MEASURES DESIGNS

2.1 Definition of the Test Statistic

The multivariate sign test based on interdirections, denoted by V_n, was proposed by Randles for the one-sample multivariate location problem. In this section we will show how this test statistic can also be applied to repeated-measures designs for detecting treatment effects.

For the one-sample multivariate location problem, we let $\underline{X}_1, \ldots, \underline{X}_n$, where $\underline{X}_i = (X_{i1}, \ldots, X_{ip})^T$, be independent and identically distributed (i.i.d.) as $\underline{X} = (X_1, \ldots, X_p)^T$, where $\underline{X}$ is from a p-dimensional absolutely continuous population with location parameter $\underline{\theta}^*$ (p×1). We would like to test

$$H_0 : \underline{\theta}^* = \underline{0} \quad \text{versus} \quad H_a: \underline{\theta}^* \neq \underline{0}.$$

Here $\underline{0}$ is used without loss of generality, since $H_0 : \underline{\theta}^* = \underline{\theta}_0$ can be tested by subtracting $\underline{\theta}_0$ from each observation $\underline{X}_i$ and testing whether these differences $(\underline{X}_i - \underline{\theta}_0)$'s are located at $\underline{0}$.

For the problem of single-factor repeated-measures designs, we let $\underline{Y}_1, \ldots, \underline{Y}_n$ be i.i.d. as $\underline{Y} = (Y_1, \ldots, Y_p)^T$, where $\underline{Y}$ is from a p-dimensional, $p \geq 2$, absolutely continuous population. Note that the components of $\underline{Y}_i$ are repeated measurements of the ith experimental unit. We will use the general mixed model with a subject by treatment interaction. In vector forms, we consider the model (see, e.g., Winer [1971], p. 278),

$$\underline{Y}_i = \beta_i \underline{1}_p + \underline{\tau} + \underline{\beta\tau}_i + \underline{\varepsilon}_i, \; i = 1, \ldots, n, \qquad (2.1.1)$$

where $\underline{1}_p$ is the p×1 vector of 1's, $\underline{\tau} = (\tau_1, \ldots, \tau_p)^T$ is the vector of fixed treatment effects, β_i represents the random effect of the ith subject, $\underline{\beta\tau}_i$ denotes the vector of the ith subject by treatments interactions, and $\underline{\varepsilon}_i$ is the vector of random error of the ith subject. We assume β_i's are i.i.d. with mean 0, $\underline{\beta\tau}_i$'s are i.i.d. with mean $\underline{0}$ and a general variance-covariance matrix possibly not H-type, described in Chapter 1, $\underline{\varepsilon}_i$'s are i.i.d. with mean $\underline{0}$ and variance-covariance matrix $\sigma_\varepsilon^2 \cdot \underline{I}_p$, where $\underline{I}_p$ is the p×p identity matrix. The random variables β_i, $\underline{\beta\tau}_i$, $\underline{\varepsilon}_i$ for i = 1, …, n are all mutually independent. We are concerned with the test of equal treatment effects, which in model (2.1.1) can be described as:

$$H_0 : \tau_1 = \tau_2 = \ldots = \tau_p \quad \text{versus} \quad H_a: \tau_j \neq \tau_{j'} \text{ for some } j \neq j'. \qquad (2.1.2)$$

Note that the sample $\underline{Y}_1, \ldots, \underline{Y}_n$ has location parameter $\underline{\tau}$. The problem is to test whether the components of the location parameter $\underline{\tau}$ are all equal. We can transform this problem to the standard one-sample multivariate location problem, described earlier, by looking at the differences among the components Y_{ij} within each observation $\underline{Y}_i$. The transformation is described below. Define

$$\underline{Z}_i = \begin{bmatrix} Y_{i1}-Y_{ip} \\ Y_{i2}-Y_{ip} \\ \cdot \\ \cdot \\ \cdot \\ Y_{i\,p-1}-Y_{ip} \end{bmatrix} = \begin{bmatrix} 1 & & & -1 \\ & 1 & & -1 \\ & & \ddots & \vdots \\ & & 1 & -1 \end{bmatrix} \begin{bmatrix} Y_{i1} \\ Y_{i2} \\ \cdot \\ \cdot \\ \cdot \\ Y_{i\,p-1} \\ Y_{ip} \end{bmatrix} = \underline{A}\,\underline{Y}_i\,,\ i = 1, \ldots, n,$$

and

$$\underline{\theta} = \begin{bmatrix} \tau_1-\tau_p \\ \tau_2-\tau_p \\ \cdot \\ \cdot \\ \cdot \\ \tau_{p-1}-\tau_p \end{bmatrix} = \begin{bmatrix} 1 & & & -1 \\ & 1 & & -1 \\ & & \ddots & \vdots \\ & & 1 & -1 \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \\ \cdot \\ \cdot \\ \cdot \\ \tau_{p-1} \\ \tau_p \end{bmatrix} = \underline{A}\,\underline{\tau}\,,$$

where

$$\underline{A} = \begin{bmatrix} 1 & & & & -1 \\ & 1 & & & -1 \\ & & \ddots & & \vdots \\ & & & 1 & -1 \end{bmatrix} \tag{2.1.3}$$

is a (p-1)×p matrix. Now, we have a (p-1)-variate sample $\underline{Z}_1, \ldots, \underline{Z}_n$, which can be modeled via

$$\underline{Z}_i = \underline{\theta} + \underline{\beta\tau}_i^* + \underline{\varepsilon}_i^* \text{ , } i = 1, \ldots, n,$$

where $\underline{\theta}$ ((p-1)×1) is the location parameter. Note that the variance-covariance matrix of $\underline{\beta\tau}_i^*$ is possibly not H-type. Thus, testing the hypotheses given in (2.1.2) is equivalent to testing

$$H_0 : \underline{\theta} = \underline{0} \quad \text{versus} \quad H_a : \underline{\theta} \neq \underline{0}. \tag{2.1.4}$$

We have shown that the test of (2.1.2) based on p-variate observations $\underline{Y}_1, \ldots, \underline{Y}_n$ can be carried out by using a multivariate location test based on a statistic like the interdirection sign statistic V_n, computed on the transformed (p-1)-variate observations $\underline{Z}_1, \ldots, \underline{Z}_n$, for testing (2.1.4). Now we will describe the test that rejects $H_0 : \underline{\theta} = \underline{0}$ for large values of the statistic

$$V_n = \frac{p-1}{n} \sum_{i=1}^{n} \sum_{k=1}^{n} \cos(\pi \widehat{p}_{ik}), \tag{2.1.5}$$

where

$$\widehat{p}_{ik} = \begin{cases} \dfrac{C_{ik}+d_n}{\binom{n}{p-2}} & \text{if } i \neq k \\[2ex] 0 & \text{if } i = k, \end{cases} \tag{2.1.6}$$

$$d_n = \frac{1}{2}\left[\binom{n}{p-2} - \binom{n-2}{p-2}\right],$$

and C_{ik} denotes the number of hyperplanes formed by the origin $\underline{0}$ and other p-2 observations (excluding $\underline{Z}_i$ and $\underline{Z}_k$) such that $\underline{Z}_i$ and $\underline{Z}_k$ are on opposite sides of the hyperplane formed. The counts $\{C_{ik} \mid 1 \leq i < k \leq n\}$, called interdirections, are used via $\pi\widehat{p}_{ik}$ to measure the angular distance between $\underline{Z}_i$ and $\underline{Z}_k$ relative to the positions of the other observations. This statistic, V_n, includes Blumen's bivariate sign test and the 2-sided univariate sign test as special cases. Also, it is affine-invariant and has a distribution-free property under H_0, for a broad class of distributions, called distributions with elliptical directions, which includes all elliptically symmetric populations and many skewed populations as well. In the next section, we will concentrate on the family of elliptically symmetric populations.

2.2 Null Distribution of V_n

In this section we will find the null distribution of V_n under the class of elliptically symmetric distributions, which is defined below.

Definition 2.2.1 Assuming the existence of a density function, the m×1 random vector $\underline{X}$ is said to have an elliptically symmetric distribution with parameters $\underline{\mu}$ (m×1) and $\underline{\Sigma}$ (m×m) if its density function is of the form

$$f_{\underline{X}}(\underline{x}) = K_m|\underline{\Sigma}|^{-1/2}h[(\underline{x} - \underline{\mu})^T\underline{\Sigma}^{-1}(\underline{x} - \underline{\mu})], \tag{2.2.2}$$

for some non-negative real-valued function h, where $\underline{\Sigma}$ is positive definite and K_m is a scalar. We will write this distribution of $\underline{X}$ as $E_m(\underline{\mu}, \underline{\Sigma})$.

Throughout this chapter, we will use $\underline{Y}_1, \ldots, \underline{Y}_n$ to denote the original sample, and use $\underline{Z}_1, \ldots, \underline{Z}_n$, defined in (2.1.3), to denote the transformed sample, to which the test statistic V_n is applied. Now, let's assume $\underline{Y}_1, \ldots, \underline{Y}_n$ are i.i.d. as $\underline{Y} = (Y_1, \ldots, Y_p)^T$, where $\underline{Y}$ is $E_p(\underline{\mu}, \underline{\Sigma})$. To apply the result of null distribution of V_n under elliptically symmetric distributions, proved by Randles, we shall first prove that the transformed sample $\underline{Z}_1, \ldots, \underline{Z}_n$ is also elliptically symmetric. To do this, we use the following lemma, which was given as an exercise in Muirhead's (1982) book.

<u>Lemma 2.2.3</u> If $\underline{X}$ is $E_m(\underline{\mu}, \underline{\Sigma})$ then :

(i) the characteristic function $\phi_{\underline{X}}(\underline{t}) = E(e^{i\underline{t}^T\underline{X}})$ has the form

$$\phi_{\underline{X}}(\underline{t}) = e^{i\underline{t}^T\underline{\mu}}\, \psi(\underline{t}^T\underline{\Sigma}\,\underline{t}) \text{ for some function } \psi, \tag{2.2.4}$$

and

(ii) provided they exist, $E(\underline{X}) = \underline{\mu}$ and $Cov(\underline{Y}) = \alpha\underline{\Sigma}$ for some constant α.

<u>Theorem 2.2.5</u> If $\underline{Y}$ is $E_p(\underline{\mu}, \underline{\Sigma})$ and $\underline{Z} = \underline{A}\,\underline{Y}$, defined in (2.1.3), then $\underline{Z}$ is $E_{p-1}(\underline{A}\,\underline{\mu}, \underline{A}\,\underline{\Sigma}\,\underline{A}^T)$.

<u>Proof of Theorem 2.2.5</u> Since $\underline{Y}$ is $E_p(\underline{\mu}, \underline{\Sigma})$, by Lemma 2.2.3, the characteristic function of $\underline{Y}$ at $\underline{t}$, a p×1 vector, has the form of

$$\phi_{\underline{Y}}(\underline{t}) = e^{i\underline{t}^T\underline{\mu}}\, \psi(\underline{t}^T\underline{\Sigma}\,\underline{t}) \text{ for some function } \psi.$$

Thus, the characteristic function of $\underline{Z}$ at $\underline{s}$, a (p-1)×1 vector, is

$$\begin{aligned} \phi_{\underline{Z}}(\underline{s}) &= E(e^{i\underline{s}^T\underline{Z}}) = E(e^{i\underline{s}^T\underline{A}\,\underline{Y}}) \\ &= E(e^{i(\underline{A}^T\underline{s})^T\underline{Y}}) \\ &= [e^{i(\underline{A}^T\underline{s})^T\underline{\mu}}]\cdot\psi[(\underline{A}^T\underline{s})^T\underline{\Sigma}(\underline{A}^T\underline{s})] \end{aligned}$$

$$= [e^{i\underline{s}^T(\underline{A}\,\underline{\mu})}]\cdot\psi[\underline{s}^T(\underline{A}\,\underline{\Sigma}\,\underline{A}^T)\underline{s}].$$

Thus, $\underline{Z}$ is $E_{p-1}(\underline{A}\,\underline{\mu}, \underline{A}\,\underline{\Sigma}\,\underline{A}^T)$.

We are now prepared to state the following theorem.

Theorem 2.2.6 Assume the observations $\underline{Y}_1, \ldots, \underline{Y}_n$ are i.i.d. from a p-variate elliptically symmetric distribution. Then, under H_0, defined in (2.1.4), V_n, defined in (2.1.5), computed on observations $\underline{Z}_1, \ldots, \underline{Z}_n$, has a small-sample distribution-free property and a limiting χ^2_{p-1} distribution.

Proof of Theorem 2.2.6 See Randles (1989), p. 1046-1047.

2.3 Asymptotic Distribution of V_n under Contiguous Alternatives

In this section we will find the asymptotic distribution of V_n under a sequence of alternatives approaching the null hypothesis $H_0 : \underline{\theta} = \underline{0}$. In doing this, we will restrict our attention to a specific class of elliptically symmetric distributions. Let's assume $\underline{Y}_1, \ldots, \underline{Y}_n$ are i.i.d. as $\underline{Y} = (Y_1, \ldots, Y_p)^T$, where $\underline{Y}$ is elliptically symmetric with a density function $f_{\underline{Y}}$ of the form

$$f_{\underline{Y}}(\underline{y}) = K_p|\underline{\Sigma}|^{-1/2}\exp\{-[(\underline{y} - \underline{\mu})^T\underline{\Sigma}^{-1}(\underline{y} - \underline{\mu})/C_0]^\nu\},\ \underline{y} \in R^p, \tag{2.3.1}$$

where

$$C_0 = \frac{p\Gamma(p/2\nu)}{\Gamma[(p+2)/2\nu]},\quad K_p = \frac{\nu\Gamma(p/2)}{\Gamma(p/2\nu)(\pi C_0)^{p/2}}, \tag{2.3.2}$$

$$\Gamma(w) = \int_0^\infty x^{w-1}e^{-x}\,dx \text{ for } w > 0,$$

and R^p is the Euclidean p-space. It can be verified that the expression in (2.3.1) is a valid density function and that $\underline{\mu}$ represents the mean and $\underline{\Sigma}$, the variance-covariance matrix. This family includes the multivariate normal distribution ($\nu = 1$), heavier-tailed distributions ($0 < \nu < 1$) and lighter-tailed distributions ($\nu > 1$). As explained in the previous sections, we will need to derive the density function of the transformed sample, whose form will be used when deriving the Pitman asymptotic relative efficiency of V_n relative to Hotelling's T^2 in next section.

Lemma 2.3.3 For the family of distributions given in expression (2.3.1), the transformed sample $\underline{Z}_1, \ldots, \underline{Z}_n$ has a density function of the form

$$f_{\underline{Z}}(\underline{z}) = K_p|\underline{A}\,\underline{\Sigma}\,\underline{A}^T|^{-1/2} g[(\underline{z}-\underline{A}\,\underline{\mu})^T(\underline{A}\,\underline{\Sigma}\,\underline{A}^T)^{-1}(\underline{z}-\underline{A}\,\underline{\mu})],\ \underline{z} \in R^{p-1}, \tag{2.3.4}$$

where

$$g(t) = \int_{-\infty}^{\infty} \exp\{-[(t+s^2)/C_0]^\nu\}ds. \tag{2.3.5}$$

Proof of Lemma 2.3.3 Letting $\underline{x} = \begin{bmatrix} \underline{z} \\ y_p \end{bmatrix}$, then we can write

$$\underline{x} = \begin{bmatrix} y_1 - y_p \\ \vdots \\ y_{p-1} - y_p \\ y_p \end{bmatrix} = \begin{bmatrix} 1 & & -1 \\ & \ddots & \vdots \\ & & -1 \\ & & 1 \end{bmatrix} \underline{y} = \begin{bmatrix} \underline{A} \\ \underline{e}_p^T \end{bmatrix} \underline{y} = \underline{B}\,\underline{y},$$

where $\underline{B} = \begin{bmatrix} \underline{A} \\ \underline{e}_p^T \end{bmatrix}$, $\underline{e}_p = (0, \ldots, 0, 1)^T$, a p×1 vector with 1 on the pth component and 0's elsewhere, and $\underline{A}$ is defined in (2.1.3). Since $\underline{B}$ is nonsingular, it follows that $\underline{y} = \underline{B}^{-1}\underline{x}$. The jacobian of the transformation, denoted by $J(\underline{y} \Rightarrow \underline{x})$, is equal to $|\underline{B}^{-1}| = |\underline{B}|^{-1} = 1$. Hence the density function of $\underline{X}$ is

$$f_{\underline{X}}(\underline{x}) = f_{\underline{Y}}(\underline{y})\,\Big|_{\,\underline{y}=\underline{B}^{-1}\underline{x}}$$

$$= K_p|\underline{\Sigma}|^{-1/2}\exp\{-[(\underline{B}^{-1}\underline{x}-\underline{\mu})^T\underline{\Sigma}^{-1}(\underline{B}^{-1}\underline{x}-\underline{\mu})/C_0]^\nu\},$$

$$= K_p|\underline{\Sigma}|^{-1/2}\exp\{-[(\underline{x}-\underline{B}\,\underline{\mu})^T(\underline{B}\,\underline{\Sigma}\,\underline{B}^T)^{-1}(\underline{x}-\underline{B}\,\underline{\mu})/C_0]^\nu\}.$$

Since $|\underline{B}\,\underline{\Sigma}\,\underline{B}^T| = |\underline{\Sigma}|$ due to $|\underline{B}| = 1$, it follows that

$$f_{\underline{X}}(\underline{x}) = K_p|\underline{B}\,\underline{\Sigma}\,\underline{B}^T|^{-1/2}\exp\{-[(\underline{x}-\underline{B}\,\underline{\mu})^T(\underline{B}\,\underline{\Sigma}\,\underline{B}^T)^{-1}(\underline{x}-\underline{B}\,\underline{\mu})/C_0]^\nu\}.$$

Letting $\underline{B}\,\underline{\Sigma}\,\underline{B}^T = \underline{V}$ and $\underline{B}\,\underline{\mu} = \underline{\eta}$, we rewrite the above expression as

$$f_{\underline{X}}(\underline{x}) = K_p|\underline{V}|^{-1/2}\exp\{-[(\underline{x}-\underline{\eta})^T\underline{V}^{-1}(\underline{x}-\underline{\eta})/C_0]^\nu\}.$$

Thus, the density function of $\underline{Z}$ is

$$f_{\underline{Z}}(\underline{z}) = \int_{-\infty}^{\infty} f_{\underline{X}}(\underline{x})\,dy_p$$

$$= K_p|\underline{V}|^{-1/2}\int_{-\infty}^{\infty}\exp\{-[(\underline{x}-\underline{\eta})^T\underline{V}^{-1}(\underline{x}-\underline{\eta})/C_0]^\nu\}dy_p. \qquad (2.3.6)$$

Note that

$$\underline{V} = \underline{B}\,\underline{\Sigma}\,\underline{B}^T = \begin{bmatrix} \underline{A}\,\underline{\Sigma}\,\underline{A}^T & \underline{A}\,\underline{\Sigma}\,\underline{e}_p \\ \underline{e}_p^T\underline{\Sigma}\,\underline{A}^T & \underline{e}_p^T\underline{\Sigma}\,\underline{e}_p \end{bmatrix} \text{ and } \underline{\eta} = \underline{B}\,\underline{\mu} = \begin{bmatrix} \underline{A}\,\underline{\mu} \\ \mu_p \end{bmatrix} = \begin{bmatrix} \underline{\theta} \\ \mu_p \end{bmatrix}.$$

Denoting $\underline{V}$ by $\begin{bmatrix} \underline{V}_{11} & \underline{V}_{12} \\ \underline{V}_{21} & V_{22} \end{bmatrix}$, we have $\underline{V}_{11} = \underline{A}\,\underline{\Sigma}\,\underline{A}^T$ is nonsingular and $V_{22} = \underline{e}_p^T\underline{\Sigma}\,\underline{e}_p$

is a positive scalar. Next we use the fact that

$$\underline{V}^{-1} = \begin{bmatrix} \underline{V}_{11} & \underline{V}_{12} \\ \underline{V}_{21} & V_{22} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \underline{V}_{11}^{-1} + \underline{V}_{11}^{-1}\underline{V}_{12}V_{22\cdot 1}^{-1}\underline{V}_{21}\underline{V}_{11}^{-1} & -\underline{V}_{11}^{-1}\underline{V}_{12}V_{22\cdot 1}^{-1} \\ -V_{22\cdot 1}^{-1}\underline{V}_{21}\underline{V}_{11}^{-1} & V_{22\cdot 1}^{-1} \end{bmatrix},$$

where

$$V_{22\cdot 1} = V_{22} - \underline{V}_{21}\underline{V}_{11}^{-1}\underline{V}_{12}.$$

We can expand the quadratic form in expression (2.3.6) as

$$\begin{aligned} & (\underline{x}-\underline{\eta})^T\underline{V}^{-1}(\underline{x}-\underline{\eta}) \\ = \quad & (\underline{z}-\underline{\theta})^T\underline{V}_{11}^{-1}(\underline{z}-\underline{\theta}) + (\underline{z}-\underline{\theta})^T\underline{V}_{11}^{-1}\underline{V}_{12}V_{22\cdot 1}^{-1}\underline{V}_{21}\underline{V}_{11}^{-1}(\underline{z}-\underline{\theta}) \\ & + (y_p-\mu_p)^2V_{22\cdot 1}^{-1} - 2(\underline{z}-\underline{\theta})^T\underline{V}_{11}^{-1}\underline{V}_{12}V_{22\cdot 1}^{-1}(y_p-\mu_p) \\ = \quad & (\underline{z}-\underline{\theta})^T\underline{V}_{11}^{-1}(\underline{z}-\underline{\theta}) + \left[(y_p-\mu_p)V_{22\cdot 1}^{-1/2}-(\underline{z}-\underline{\theta})^T\underline{V}_{11}^{-1}\underline{V}_{12}V_{22\cdot 1}^{-1/2}\right]^2. \end{aligned} \tag{2.3.7}$$

Using the fact that

$$|\underline{V}| = |\underline{V}_{11}|\cdot|V_{22} - \underline{V}_{21}\underline{V}_{11}^{-1}\underline{V}_{12}| = |\underline{V}_{11}|\cdot|V_{22\cdot 1}|$$

and expression (2.3.7), we may write expression (2.3.6) as

$$\begin{aligned} & f_{\underline{Z}}(\underline{z}) \\ = \quad & K_p\,|\underline{V}_{11}|^{-1/2}\cdot|V_{22\cdot 1}|^{-1/2} \\ & \cdot \int_{-\infty}^{\infty} \exp\{-[((\underline{z}-\underline{\theta})^T\underline{V}_{11}^{-1}(\underline{z}-\underline{\theta}) + ((y_p-\mu_p)V_{22\cdot 1}^{-1/2} - (\underline{z}-\underline{\theta})^T\underline{V}_{11}^{-1}\underline{V}_{12}V_{22\cdot 1}^{-1/2})^2)/C_0]^\nu\}\,dy_p. \end{aligned}$$

Taking $s = (y_p-\mu_p)V_{22\cdot 1}^{-1/2} - (\underline{z}-\underline{\theta})^T\underline{V}_{11}^{-1}\underline{V}_{12}V_{22\cdot 1}^{-1/2}$, we can write the above expression as

$$f_{\underline{Z}}(\underline{z}) = K_p |\underline{V}_{11}|^{-1/2} \int_{-\infty}^{\infty} \exp\{-[((\underline{z}-\underline{\theta})^T \underline{V}_{11}^{-1}(\underline{z}-\underline{\theta}) + s^2)/C_0]^v\} ds. \qquad (2.3.8)$$

Since $\underline{V}_{11} = \underline{A}\,\underline{\Sigma}\,\underline{A}^T$ and $\underline{\theta} = \underline{A}\,\underline{\mu}$, expression (2.3.8) is equivalent to expression (2.3.4). This completes the proof. It can be verified that expression (2.3.8) is valid density function.

Now, we are in the position to discuss the asymptotic distribution of V_n under a sequence of alternatives approaching the null distribution. Under H_0, $\underline{Z}_1, \ldots, \underline{Z}_n$ are i.i.d. with density of the form

$$f_{\underline{Z}}(\underline{z}) = K_p |\underline{A}\,\underline{\Sigma}\,\underline{A}^T|^{-1/2} \int_{-\infty}^{\infty} \exp\{-[(\underline{z}^T(\underline{A}\,\underline{\Sigma}\,\underline{A}^T)^{-1}\underline{z}+s^2)/C_0]^v\} ds. \qquad (2.3.9)$$

Under a sequence of alternatives let $\underline{Z}_1, \ldots, \underline{Z}_n$ be i.i.d. with density $f_{\underline{Z}}(\underline{z}-\underline{c}n^{-1/2})$, where $f_{\underline{Z}}$ is given in (2.3.9) and $\underline{c} \in R^{p-1}-\{0\}$ is arbitrary, but fixed. It is shown, with the outline of the proof given in Appendix A, that if $4v + p > 3$, then

$$0 < I_{\underline{c}}(f_{\underline{Z}}) = \int \left[\frac{\underline{c}^T \partial f_{\underline{Z}}(\underline{z})}{f_{\underline{Z}}(\underline{z})} \right]^2 f_{\underline{Z}}(\underline{z}) d\underline{z} < \infty \quad \text{for all } \underline{c} \neq \underline{0}.$$

Thus, when $4v + p > 3$, the rationale of Hájek and Sidák (1967), p. 212-213 and earlier, shows that the alternatives are contiguous to the null hypothesis. Noting that both V_n and Hotelling's T^2 are affine-invariant, we can, without loss of generality, assume throughout that $\underline{A}\,\underline{\Sigma}\,\underline{A}^T = \underline{I}_{p-1}$, the (p-1)×(p-1) identity matrix.

Theorem 2.3.10 Let $\underline{Y}_1, \ldots, \underline{Y}_n$ be i.i.d. with density function $f_{\underline{Y}}$ given in expression (2.3.1). Assume the density function of $\underline{Z}_1, \ldots, \underline{Z}_n$ has the form of

$$f_{\underline{Z}}(\underline{z}) = K_p g(\underline{z}^T\underline{z}) = K_p \int_{-\infty}^{\infty} \exp\{-[(\underline{z}^T\underline{z}+s^2)/C_0]^v\} ds, \ \underline{z} \in R^{p-1}.$$

If $4v + p > 3$, then under the sequence of contiguous alternatives,

$$V_n \xrightarrow{d} \chi^2_{p-1}\left(\frac{4}{p-1}\left\{E_{H_0}\left[\frac{Rg'(R^2)}{g(R^2)}\right]\right\}^2 \underline{c}^T\underline{c}\right), \tag{2.3.11}$$

where

$$g(R^2) = \int_{-\infty}^{\infty} \exp\{-[(R^2+s^2)/C_0]^v\}ds,$$

and

$$g'(R^2) = dg(R^2)/dR^2 \text{ and } R^2 = \underline{Z}^T\underline{Z}.$$

Proof of Theorem 2.3.10 See Randles (1989), p. 1050.

Note that the noncentrality parameter in (2.3.11) can be simplified and this will be done in the next section.

2.4 The Pitman Asymptotic Relative Efficiency of V_n Relative to Hotelling's T^2

In this section we will use Pitman relative efficiencies to make comparisons between V_n and Hotelling's T^2in the repeated measurement design settings. Because these statistics are all affine-invariant we may, without loss of generality, make the simplifying assumption that the transformed sample variance-covariance matrix is the identity. We will apply the asymptotic results under the contiguous alternatives in Theorem 2.3.10.

The Pitman approach to asymptotic relative efficiency compares two test sequences $\{S_{n_i}\}$ and $\{T_{m_i}\}$ as the sequence of alternatives $H_i : \underline{\theta} = \underline{\theta}_i$, approaches the null, which we are taken to be $H_0 : \underline{\theta} = \underline{0}$. The subscripts n_i and m_i are the sample sizes for tests S_{n_i} and T_{m_i}, respectively. Let $\beta\{S_{n_i}, \underline{\theta}_i\}$ and $\beta\{T_{m_i}, \underline{\theta}_i\}$ denote the powers of the tests based on

S_{n_i} and T_{m_i}, respectively, when $\underline{\theta} = \underline{\theta}_i$. Assume that n_i and m_i are such that the two sequences of tests have the same limiting significance level α and

$$\alpha < \lim_{i\to\infty} \beta\{S_{n_i}, \underline{\theta}_i\} = \lim_{i\to\infty} \beta\{T_{m_i}, \underline{\theta}_i\} < 1.$$

Then the Pitman asymptotic relative efficiency (ARE) of $\{S_{n_i}\}$ relative to $\{T_{m_i}\}$ (or simply of S relative to T) is

$$\text{ARE}(S, T) = \lim_{i\to\infty} \frac{m_i}{n_i},$$

provided the limit exists and is the same for all such sequences $\{n_i\}$ and $\{m_i\}$, and independent of the $\{\underline{\theta}_i\}$ sequence. (See, e.g., Randles and Wolfe [1979], p. 144.)

Hannan (1956) shows that if, under the sequence of alternatives H_i, the test sequences $\{S_{n_i}\}$ and $\{T_{m_i}\}$ are asymptotically noncentral Chi-square with the same degrees of freedom and noncentrality parameters, δ_S^2 and δ_T^2, respectively, then

$$\text{ARE}(S, T) = \frac{\delta_S^2}{\delta_T^2}.$$

It's well known that under the sequence of contiguous alternatives described in the last section, and taking $\underline{A}\,\Sigma\,\underline{A}^T = \underline{I}_{p-1}$, the asymptotic distribution of Hotelling's T^2, where $T^2 = n\,\overline{\underline{Z}}^T (\hat{\underline{\Sigma}})^{-1}\overline{\underline{Z}}$ with $\overline{\underline{Z}} = n^{-1} \sum_{i=1}^{n} \underline{Z}_i$ and $\hat{\underline{\Sigma}} = (n-1)^{-1} \sum_{i=1}^{n} (\underline{Z}_i-\overline{\underline{Z}})(\underline{Z}_i-\overline{\underline{Z}})^T$, is noncentral chi-square with p-1 degrees of freedom and noncentrality parameter

$$\delta_{T^2}^2 = \underline{c}^T\underline{c}. \tag{2.4.1}$$

(See, e.g., Puri and Sen [1971], p. 173.) To derive $\text{ARE}(V_n, T^2)$ we use the following lemma.

Lemma 2.4.2 Let $\underline{Y}_1, \ldots, \underline{Y}_n$ be i.i.d. with density function given in expression (2.3.1), and $R^2 = \underline{Z}^T\underline{Z}$. Taking $\underline{A}\,\underline{\Sigma}\,\underline{A}^T = \underline{I}_{p-1}$, then, under $H_0 : \underline{\theta} = \underline{0}$, the density function of R^2 is of the form

$$f_{R^2}(r) = \frac{K_p \pi^{(p-1)/2}}{\Gamma[(p-1)/2]} r^{(p-1)/2-1} g(r),\ r > 0, \tag{2.4.3}$$

where

$$g(r) = \int_{-\infty}^{\infty} \exp\{-[(r+s^2)/C_0]^{\nu}\}ds, \tag{2.4.4}$$

and K_p and C_0 are defined in (2.3.2).

Proof of Lemma 2.4.2 Taking $\underline{A}\,\underline{\Sigma}\,\underline{A}^T = \underline{I}_{p-1}$ and under $H_0 : \underline{\theta} = \underline{0}$, we have $\underline{Z}$ is $E_{p-1}(\underline{0}, \underline{I}_{p-1})$ with density function of the form

$$f_{\underline{Z}}(\underline{z}) = K_p g(\underline{z}^T\underline{z}) = K_p \int_{-\infty}^{\infty} \exp\{-[(\underline{z}^T\underline{z}+s^2)/C_0]^{\nu}\}ds,\ \underline{z} \in R^{p-1}.$$

Thus, $R^2 = \underline{Z}^T\underline{Z}$ has density of the form given in expression (2.4.3). (See, e.g., Muirhead [1982], p. 37.)

Theorem 2.4.5 Assume $\underline{Y}_1, \ldots, \underline{Y}_n$ are i.i.d. from a density given by (2.3.1). In the repeated measurement settings, taking $\underline{A}\,\underline{\Sigma}\,\underline{A}^T = \underline{I}_{p-1}$, if $4\nu + p > 3$, then the Pitman asymptotic relative efficiency of V_n relative to Hotelling's T^2 is

$$ARE(V_n,\ T^2) = \frac{4\Gamma^4(p/2)\Gamma^2[(p-1)/2\nu]\Gamma[(p+2)/2\nu]}{p(p-1)\Gamma^4[(p-1)/2]\Gamma^3(p/2\nu)},\ p \geq 2. \tag{2.4.6}$$

Proof of Theorem 2.4.5 If $4\nu + p > 3$, taking $\underline{A}\,\underline{\Sigma}\,\underline{A}^T = \underline{I}_{p-1}$ and using expression (2.3.11), we have, under the sequence of contiguous alternatives, the test V_n is

asymptotically noncentral chi-square with p-1 degrees of freedom and noncentrality parameter $\delta^2_{V_n}$, where

$$\delta^2_{V_n} = \frac{4}{p\text{-}1}\left\{E_{H_0}\left[\frac{Rg'(R^2)}{g(R^2)}\right]\right\}^2 \underline{c}^T\underline{c}\,.$$

It follows that

$$\mathrm{ARE}(V_n,\ T^2) = \frac{\delta^2_{V_n}}{\delta^2_{T^2}} = \frac{4}{p\text{-}1}\left\{E_{H_0}\left[\frac{Rg'(R^2)}{g(R^2)}\right]\right\}^2. \tag{2.4.7}$$

Under H_0, we find, using expressions (2.4.3) and (2.4.4),

$$E\left[\frac{R\cdot g'(R^2)}{g(R^2)}\right]$$

$$= E\left[\frac{\sqrt{R^2}\cdot g'(R^2)}{g(R^2)}\right]$$

$$= \int_0^\infty \frac{\sqrt{r}\cdot g'(r)}{g(r)} f_{R^2}(r)dr$$

$$= \int_0^\infty \frac{\sqrt{r}\cdot g'(r)}{g(r)} \frac{K_p\pi^{(p-1)/2}}{\Gamma((p-1)/2)} r^{(p-1)/2-1} g(r)dr$$

$$= \left(\frac{K_p\pi^{(p-1)/2}}{\Gamma((p-1)/2)}\right)\int_0^\infty r^{p/2-1} g'(r)dr\,.$$

It can be verified that

$$g'(r) = \left(\int_{-\infty}^{\infty} \exp\{-[(r+s^2)/C_0]^\nu\}ds\right)'$$

$$= \int_{-\infty}^{\infty} \left[\frac{\partial(\exp\{-[(r+s^2)/C_0]^\nu\})}{\partial r}\right] ds.$$

(See, e.g., Trench [1978], Theorem 5.2 and 5.6 on p. 581 and p. 586, respectively.)

Letting

$$C_p = \frac{K_p \pi^{(p-1)/2}}{\Gamma[(p-1)/2]}, \tag{2.4.8}$$

we have

$$E\left[\frac{R \cdot g'(R^2)}{g(R^2)}\right]$$

$$= C_p \int_0^\infty r^{p/2-1} \int_{-\infty}^\infty \left[\frac{\partial(\exp\{-[(r+s^2)/C_0]^\nu\})}{\partial r}\right] ds\, dr$$

$$= C_p \int_0^\infty r^{p/2-1} \int_{-\infty}^\infty \exp\{-[(r+s^2)/C_0]^\nu\}(-\nu)((r+s^2)/C_0)^{\nu-1} C_0^{-1} ds\, dr$$

$$= \left(\frac{-2\nu C_p}{C_0^\nu}\right) \int_0^\infty \int_0^\infty r^{p/2-1} \exp\{-[(r+s^2)/C_0]^\nu\}(r+s^2)^{\nu-1} ds\, dr.$$

Taking $r = t^2$, we can write

$$E\left[\frac{R \cdot g'(R^2)}{g(R^2)}\right] = \left(\frac{-4\nu C_p}{C_0^\nu}\right) \int_0^\infty \int_0^\infty t^{p-1} \exp\{-[(t^2+s^2)/C_0]^\nu\}(t^2+s^2)^{\nu-1} ds\, dt. \tag{2.4.9}$$

Letting $s = \sqrt{C_0}\, x^{1/\nu}\sin(\theta)$ and $t = \sqrt{C_0}\, x^{1/\nu}\cos(\theta)$, we find that

$$\partial s/\partial\theta = \sqrt{C_0}\, x^{1/\nu}\cos(\theta),\ \ \partial s/\partial x = \nu^{-1}\sqrt{C_0}\, x^{(1-\nu)/\nu}\sin(\theta),$$

$$\partial t/\partial\theta = -\sqrt{C_0}\, x^{1/\nu}\sin(\theta),\ \ \partial t/\partial x = \nu^{-1}\sqrt{C_0}\, x^{(1-\nu)/\nu}\cos(\theta),$$

and

$$s^2+t^2 = C_0\, x^{2/\nu}.$$

The jacobian of the transformation is

$$v^{-1}C_0\,x^{(2-v)/v}\cos^2(\theta) + v^{-1}C_0\,x^{(2-v)/v}\sin^2(\theta) = v^{-1}C_0\,x^{(2-v)/v}.$$

Thus, we can write (2.4.9) as

$$E\left[\frac{R\cdot g'(R^2)}{g(R^2)}\right]$$

$$= \left(\frac{-4v\cdot C_p}{C_0^v}\right)\int_0^{\infty}\int_0^{\pi/2}(\sqrt{C_0}\,x^{1/v}\cos(\theta))^{p-1}(C_0\,x^{2/v})^{v-1}\exp(-x^2)v^{-1}C_0\,x^{(2-v)/v}\,d\theta\,dx$$

$$= \left(-4C_pC_0^{(p-1)/2}\right)\cdot\left(\int_0^{\infty}x^{(p+v-1)/v}\exp(-x^2)dx\right)\cdot\left(\int_0^{\pi/2}\cos^{p-1}(\theta)d\theta\right). \qquad (2.4.10)$$

The constant term in the product above is

$$-4C_pC_0^{(p-1)/2}$$

$$= -4\frac{K_p\pi^{(p-1)/2}}{\Gamma((p-1)/2)}C_0^{(p-1)/2} \text{ (using expression (2.4.8))}$$

$$= -4\frac{v\Gamma(p/2)}{\Gamma(p/2v)(\pi C_0)^{p/2}}\cdot\frac{\pi^{(p-1)/2}}{\Gamma[(p-1)/2]}C_0^{(p-1)/2} \text{ (using expression (2.3.2))}$$

$$= -4\frac{v\Gamma(p/2)}{\Gamma(p/2v)\Gamma[(p-1)/2]}\cdot\frac{1}{\sqrt{\pi}\sqrt{C_0}}.$$

The first integral in expression (2.4.10) is

$$\int_0^{\infty}x^{(p+v-1)/v}\exp(-x^2)dx$$

$$= \frac{1}{2}\int_0^{\infty}y^{(p-1)/2v}\exp(-y)dy \text{ (using } y = x^2\text{)}$$

$$= \frac{1}{2}\int_0^\infty y^{(p+2v-1)/2v-1}\exp(-y)dy$$

$$= \frac{1}{2}\Gamma(\frac{p+2v-1}{2v}) = \frac{1}{2}\Gamma(\frac{p-1}{2v}+1)$$

$$= \frac{1}{2}\,\frac{(p-1)}{2v}\,\Gamma(\frac{p-1}{2v}) = \frac{(p-1)}{4v}\,\Gamma(\frac{p-1}{2v}).$$

The second integral in (2.4.10) is

$$\int_0^{\pi/2}\cos^{p-1}(\theta)d\theta = \int_0^{\pi/2}\sin^{p-1}(\theta)d\theta = \frac{\sqrt{\pi}\,\Gamma(\frac{p}{2})}{2\Gamma(\frac{p+1}{2})}.$$

(See, e.g., Beyer [1987], p. 289.) Thus, the expected value in expression (2.4.10) is

$$E\Big[\frac{R\cdot g'(R^2)}{g(R^2)}\Big]$$

$$= \Big[-4\,\frac{v\Gamma(p/2)}{\Gamma(p/2v)\Gamma((p-1)/2)}\,\frac{1}{\sqrt{\pi}\sqrt{C_0}}\Big]\cdot\Big[\frac{(p-1)}{4v}\,\Gamma(\frac{p-1}{2v})\Big]\cdot\Big[\frac{\sqrt{\pi}\,\Gamma(\frac{p}{2})}{2\Gamma(\frac{p+1}{2})}\Big]$$

$$= \frac{-(p-1)\,\Gamma^2(\frac{p}{2})\,\Gamma(\frac{p-1}{2v})}{2\Gamma(\frac{p}{2v})\Gamma(\frac{p-1}{2})\Gamma(\frac{p+1}{2})\sqrt{C_0}}. \qquad (2.4.11)$$

Therefore, using expressions (2.4.7) and (2.4.11), we have

$$\mathrm{ARE}(V_n, T^2) = \frac{4}{p-1}\Big[\frac{-(p-1)\,\Gamma^2(\frac{p}{2})\,\Gamma(\frac{p-1}{2v})}{2\Gamma(\frac{p}{2v})\Gamma(\frac{p-1}{2})\Gamma(\frac{p+1}{2})\sqrt{C_0}}\Big]^2$$

$$= \Big[\frac{(p-1)\,\Gamma^4(\frac{p}{2})\,\Gamma^2(\frac{p-1}{2v})}{\Gamma^2(\frac{p}{2v})\Gamma^2(\frac{p-1}{2})\Gamma^2(\frac{p+1}{2})}\Big]\cdot\Big[\frac{\Gamma(\frac{p+2}{2v})}{p\Gamma(\frac{p}{2v})}\Big]. \text{ (using (2.3.2))}$$

Noting that

$$\Gamma(\tfrac{p+1}{2}) = \Gamma[(\tfrac{p-1}{2})+1] = (\tfrac{p-1}{2})\Gamma(\tfrac{p-1}{2}),$$

we have

$$\begin{aligned} ARE(V_n, T^2) &= \frac{(p-1)\,\Gamma^4(\frac{p}{2})\,\Gamma^2(\frac{p-1}{2v})\,\Gamma(\frac{p+2}{2v})}{p\Gamma^3(\frac{p}{2v})\Gamma^4(\frac{p-1}{2})(\frac{p-1}{2})^2} \\ &= \frac{4\,\Gamma^4(\frac{p}{2})\,\Gamma^2(\frac{p-1}{2v})\,\Gamma(\frac{p+2}{2v})}{p(p-1)\,\Gamma^3(\frac{p}{2v})\Gamma^4(\frac{p-1}{2})} . \end{aligned}$$

This ARE is evaluted in Table 2.1 for selected values of v and p satisfying the condition $4v + p > 3$. Note that when $p \geq 3$, $4v + p > 3$ for all $v > 0$, and when $p = 2$, $4v + p > 3$ for all $v > .25$. When the underlying population is multivariate normal or close to normal (v = 1.0, .75, respectively), Hotelling's T^2 performs better, yet V_n appears to be quite competitive and the efficiencies increase as p increases. For light-tailed distributions (v = 2, 3, 4, and 5) the sign test is not as effective as T^2, but the ARE's increase with p. For heavy-tailed distributions (v = .50, .25, .20, .15, and .10) V_n is more effective than T^2. Here the efficiencies decrease with p. In fact, it can be verified, using Stirling's formula for approximation of m! and the result

$$\lim_{x\to\infty} (1+\lambda x)^{1/x} = e^{\lambda} \text{ for constant } \lambda,$$

that for fixed v, ARE $\to 1$ as $p \to \infty$. For fixed p, it can be shown that as $v \to 0$, the ARE $\to \infty$ and as $v \to \infty$, the ARE $\to \{4p^2\Gamma^4(p/2)\}/\{(p+2)(p-1)^3\Gamma^4((p-1)/2)\}$, the latter is evaluated for p = 2 to 12 and is displayed in Table 2.1 under the column $v = \infty$.

Table 2.1

ARE (V_n, T^2)

p	ν: ∞	5.0	4.0	3.0	2.0	1.0	.75	.50	.25	.20	.15	.10
2	0.4053	0.4205	0.4278	0.4422	0.4784	0.6366	0.7851	1.2159	4.7283*	9.4001*	29.6622*	297.1379*
3	0.5552	0.5738	0.5822	0.5983	0.6364	0.7854	0.9110	1.2337	3.1089	4.9469	10.7403	50.7101
4	0.6404	0.6599	0.6682	0.6837	0.7191	0.8488	0.9519	1.2008	2.4262	3.4521	6.2165	20.1730
5	0.6971	0.7164	0.7203	0.7387	0.7708	0.8836	0.9700	1.1711	2.0674	2.7484	4.4185	11.4238
6	0.7378	0.7566	0.7640	0.7773	0.8063	0.9054	0.9796	1.1477	1.8496	2.3487	3.4980	7.7606
7	0.7687	0.7868	0.7937	0.8059	0.8323	0.9204	0.9852	1.1295	1.7043	2.0939	2.9512	5.8638
8	0.7930	0.8103	0.8167	0.8280	0.8521	0.9313	0.9888	1.1151	1.6008	1.9182	2.5934	4.7408
9	0.8126	0.8292	0.8351	0.8456	0.8677	0.9396	0.9912	1.1035	1.5235	1.7903	2.3428	4.0123
10	0.8287	0.8446	0.8502	0.8600	0.8804	0.9461	0.9929	1.0939	1.4636	1.6931	2.1583	3.5075
11	0.8423	0.8575	0.8628	0.8719	0.8908	0.9513	0.9942	1.0860	1.4160	1.6169	2.0173	3.1399
12	0.8539	0.8685	0.8734	0.8819	0.8996	0.9556	0.9951	1.0793	1.3771	1.5557	1.9061	2.8619

*Conjectured ARE's when p = 2 and ν ≤ .25, using the form of the efficiency given in Theorem 2.4.5.

CHAPTER 3
A MULTIVARIATE SIGNED-RANK TEST BASED ON INTERDIRECTIONS FOR REPEATED-MEASURES DESIGNS

3.1 Definition of the Test Statistic

In this section we describe the multivariate signed-rank test, denoted by W_n, proposed by Peters and Randles (1990) for the one-sample multivariate location problem. As explained in section 2.1, this test can also be applied to repeated measures designs for detecting treatment effects.

Using the same notations as in chapter 2, we let $\underline{Y}_1, \ldots, \underline{Y}_n$ be i.i.d. from a p-variate elliptically symmetric distribution and $\underline{Z}_1, \ldots, \underline{Z}_n$ be the transformed sample defined in (2.1.3). Recalling the result of Theorem 2.2.5, we have that $\underline{Z}_1, \ldots, \underline{Z}_n$ are i.i.d. from a (p-1)-variate elliptically symmetric distribution. Thus, it is logical to measure the distance of each observation $\underline{Z}_i$, i = 1, ..., n, from the origin in terms of elliptical contours and to use the ranks of these distances along with the observations' directions in forming a test statistic.

We now describe such a signed-rank statistic based on V_n which includes the univariate signed-rank statistic as a special case. Specifically, let us form estimated Mahalanobis distances via

$$\hat{D}_i = \underline{Z}_i^T \hat{\underline{\Sigma}}^{-1} \underline{Z}_i \text{ , } i=1,\ldots, n, \qquad (3.1.1)$$

where

$$\hat{\underline{\Sigma}} = \frac{1}{n} \sum_{i=1}^{n} \underline{Z}_i \underline{Z}_i^T$$

is a consistent estimator of the null hypothesis variance-covariance matrix of $\underline{Z}_i$, provided it exists, with H_0 defined in (2.1.4). Let Q_i = Rank ($\hat{D}_i$) among $\hat{D}_1, \ldots, \hat{D}_n$, i = 1, ..., n. We now weight the (i, k)th term in the sum V_n, defined in (2.1.5), by Q_iQ_k, and consider the statistic

$$W_n = \frac{3(p-1)}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{n} \cos(\pi \hat{p}_{ik}) \frac{Q_i}{n} \frac{Q_k}{n}, \tag{3.1.2}$$

where $\hat{p}_{ik}$ is defined in (2.1.6). We reject H_0 in favor of H_a for large values of the statistic W_n.

Since the $\hat{p}_{ik}$, i, k = 1, ..., n, are invariant with respect to a nonsingular linear transformation (as shown by Randles [1989]) as are the $\hat{D}_i$, i = 1, ..., n, it is clear that W_n is likewise affine-invariant. When p = 2, the test based on W_n is the two-sided univariate Wilcoxon signed-rank test. For p > 2, W_n does not have a small-sample distribution-free property, but its large-sample null distribution is convenient, as is shown in the next section.

3.2 Asymptotics

In this section we develop the asymptotic distributional properties of nW_n under the class of elliptically symmetric distributions. Assume that $\underline{Y}$ is elliptically symmetric, that is, the density function of $\underline{Y}$ is of the form

$$f_{\underline{Y}}(\underline{y}) = K_p|\underline{\Sigma}|^{-1/2} h[(\underline{y} - \underline{\mu})^T \underline{\Sigma}^{-1}(\underline{y} - \underline{\mu})], \ \underline{y} \in R^p, \tag{3.2.1}$$

where $\underline{\mu}$ is the point of symmetry, $\underline{\Sigma}$ is the variance-covariance matrix of $\underline{Y}$, provided it exists, and $K_p > 0$. The null distribution of nW_n is established in the following lemma.

<u>Theorem 3.2.2</u> Assume the observations $\underline{Y}_1, \ldots, \underline{Y}_n$ are i.i.d. from a p-variate elliptically symmetric distribution with a density function defined in (3.2.1). Then, in the repeated measurement settings, the test nW_n, defined in (3.1.2), has a limiting χ^2_{p-1} distribution under H_0, defined in (2.1.4).

<u>Proof of Theorem 3.2.2</u> Note that, under H_0, taking $\underline{A}\,\underline{\Sigma}\,\underline{A}^T = \underline{I}_{p-1}$, the (p-1)×(p-1) identity matrix, $\underline{Z}_1, \ldots, \underline{Z}_n$ are i.i.d. as $\underline{Z} = (Z_1, \ldots, Z_{p-1})$, where $\underline{Z}$ is from an elliptically symmetric distribution and can be expressed as $\underline{Z} = R\underline{U}$, where R^2, as before, equals $\underline{Z}^T\underline{Z}$ and $\underline{U}$ is distributed uniformly on the (p-1)-dimensional unit-sphere independent of R. (See, e.g., Johnson [1987].) It can be verified that $E[(Z_jZ_k)^2] = E[R^4U_j^2U_k^2] = E[R^4]E[U_j^2U_k^2] \le E[R^4] < \infty$, for all j, k = 1, …, p-1. Thus, via the Lindeberg-Levy Central Limit Theorem (see, e.g., Serfling [1980], p. 28), each element of $\sqrt{n}(\hat{\underline{\Sigma}} - \underline{I}_{p-1})$, where $\hat{\underline{\Sigma}} = \frac{1}{n}\sum_{i=1}^{n} \underline{Z}_i\,\underline{Z}_i^T$, is asymptotic normal under H_0. Therefore, we have $\sqrt{n}(\hat{\underline{\Sigma}} - \underline{I}_{p-1}) = O_p(1)$ under H_0. (See, e.g., Serfling [1980], p. 8.) So, the test nW_n has a limiting χ^2_{p-1} distribution under H_0. (See Peters and Randles [1990], p. 553.)

Next, we derive the Pitman asymptotic relative efficiency of nW_n relative to Hotelling's T^2. To do this, we first establish the asymptotic distribution of nW_n under contiguous alternatives, as described in section 2.3, for a general class of elliptically symmetric distributions.

<u>Lemma 3.2.3</u> Suppose $\underline{X}_1, \ldots, \underline{X}_n$ are i.i.d. from an elliptically symmetric distribution with density of the form

$$f_{\underline{X}}(\underline{x}) = K_m h[(\underline{x} - \underline{\mu})^T(\underline{x} - \underline{\mu})],\ \ \underline{x} \in R^m,$$

satisfying

$$0 < I_{\underline{d}}(f_{\underline{X}}) = \int \Big[\frac{\underline{d}^T \partial f_{\underline{X}}(\underline{x})}{f_{\underline{X}}(\underline{x})} \Big]^2 f_{\underline{X}}(\underline{x}) d\underline{x} < \infty \quad \text{for all } \underline{d} \in R^m\text{-}\{\underline{0}\}. \tag{3.2.4}$$

Under the sequence of contiguous alternatives for which $\underline{X}$ has density of the form $f_{\underline{X}}(\underline{x}\text{-}\underline{d}/\sqrt{n})$, we have

$$nW_n \xrightarrow{d} \chi^2_m \Big(\tfrac{12}{m} \Big\{ E_{H_0}\Big[\frac{K(R^2)Rh'(R^2)}{h(R^2)}\Big] \Big\}^2 \underline{d}^T \underline{d} \Big), \tag{3.2.5}$$

where

H_0 implies $\underline{\mu} = \underline{0}$, $R^2 = \underline{X}^T \underline{X}$, $h'(R^2) = dh(R^2)/dR^2$, and $K(t) = P_{H_0}(R^2 \le t)$, $t \ge 0$.

Proof of Lemma 3.2.3 Let

$$T_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Big[\frac{\underline{d}^T \partial f_{\underline{X}}(\underline{X}_i)}{f_{\underline{X}}(\underline{X}_i)} \Big],$$

$$\underline{Y}^*_n = \sqrt{3m} \Big[\frac{1}{\sqrt{n}} \sum_{i=1}^{n} U_{i1} K(R_i^2), \ldots, \frac{1}{\sqrt{n}} \sum_{i=1}^{n} U_{im} K(R_i^2) \Big]^T$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sqrt{3m}\, \underline{U}_i K(R_i^2),$$

and

$$S_n = \underline{\lambda}^T \underline{Y}^*_n,$$

where $\underline{d}, \underline{\lambda} \in R^m\text{-}\{\underline{0}\}$, and $\underline{U}_i = [U_{i1}, \ldots, U_{im}]^T$ is distributed uniformly on the m-dimensional unit-sphere independent of R_i. Here $\underline{X}_i = R_i \underline{U}_i$ and $R_i^2 = \underline{X}_i^T \underline{X}_i$, and we can write

$$T_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{\underline{d}^T K_m h'(\underline{X}_i^T \underline{X}_i)\cdot 2\underline{X}_i}{K_m h(\underline{X}_i^T \underline{X}_i)}\right]$$

$$= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{2h'(R_i^2)\cdot R_i \underline{d}^T \underline{U}_i}{h(R_i^2)}\right].$$

Thus, under H_0,

$$\begin{bmatrix} S_n \\ T_n \end{bmatrix} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\begin{bmatrix} \sqrt{3m}\, K(R_i^2)\underline{\lambda}^T \underline{U}_i \\ 2h'(R_i^2)\cdot R_i \underline{d}^T \underline{U}_i / h(R_i^2) \end{bmatrix} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\begin{bmatrix} V_i \\ W_i \end{bmatrix} \quad \text{(say)},$$

where

$$V_i = \sqrt{3m}\, K(R_i^2)\underline{\lambda}^T \underline{U}_i \quad \text{and} \quad W_i = 2h'(R_i^2)\cdot R_i \underline{d}^T \underline{U}_i / h(R_i^2).$$

Note that, under H_0, $\begin{bmatrix} V_i \\ W_i \end{bmatrix}$'s are i.i.d. with

$$E\begin{bmatrix} V_i \\ W_i \end{bmatrix} = \underline{0} \quad \text{and} \quad \mathrm{Var}\begin{bmatrix} V_i \\ W_i \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix},$$

where

$$\sigma_{11} = E(V_i^2) = 3mE[K^2(R_i^2)]\cdot E(\underline{\lambda}^T \underline{U}_i \underline{U}_i^T \underline{\lambda}) = \underline{\lambda}^T \underline{\lambda}$$

since

$$E[K^2(R_i^2)] = \frac{1}{3} \text{ and } E(\underline{U}_i \underline{U}_i^T) = \frac{1}{m} I_m,$$

$$\sigma_{22} = E(W_i^2) = \frac{4}{m}\left\{ E\left[\frac{h'(R_i^2)\cdot R_i}{h(R_i^2)}\right]\right\}^2 \underline{d}^T \underline{d},$$

and

$$\sigma_{12} \;=\; E(V_iW_i) = \frac{2\sqrt{3}}{\sqrt{m}} E\left[\frac{K(R_i^2)R_ih'(R_i^2)}{h(R_i^2)} \right] \underline{\lambda}^T\underline{d}.$$

Thus,

$$\begin{bmatrix} S_n \\ T_n \end{bmatrix} \xrightarrow{d} N_2\left(\underline{0}, \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \right) \text{ under } H_0.$$

(See, e.g., Serfling [1980], Theorem B, p. 28.) Applying LeCam's Theorems on contiguity (see Hájek and Sidák [1967], p. 212-213), under the condition defined in (3.2.4), we have

$$S_n \xrightarrow{d} N(\sigma_{12}, \sigma_{11})$$

under the sequence of alternatives. Therefore, under the sequence of alternatives,

$$\underline{Y}_n^* \xrightarrow{d} N_m\left(\frac{2\sqrt{3}}{\sqrt{m}} E_{H_0}\left[\frac{K(R^2)Rh'(R^2)}{h(R^2)} \right] \underline{d},\ \underline{I}_m \right),$$

and hence

$$nW_n \xrightarrow{d} \chi_m^2\left(\frac{12}{m} \left\{ E_{H_0}\left[\frac{K(R^2)Rh'(R^2)}{h(R^2)} \right] \right\}^2 \underline{d}^T\underline{d} \right).$$

(See Peters and Randles [1990], Result 1 and Theorem 2, on p. 555 and p. 556, respectively.)

Theorem 3.2.6 Assume $\underline{Y}_1, \ldots, \underline{Y}_n$ are i.i.d. from a density given by (2.3.1). In the repeated measurement settings, taking $\underline{A}\,\Sigma\,\underline{A}^T = \underline{I}_{p-1}$, if $4v + p > 3$, then the Pitman asymptotic relative efficiency of nW_n relative to Hotelling's T^2 is

$$\mathrm{ARE}(nW_n,\ T^2) = \frac{12}{p-1}\left(\frac{16\nu\, C_p^2}{C_0^{\nu}}\right)^2 .$$

$$\left\{\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{r} s^{p-2} r^{p-1}\,(r^2+t^2)^{\nu-1}\ \exp\{-[(s^2+u^2)/C_0]^{\nu}\}\exp\{-[(r^2+t^2)/C_0]^{\nu}\}ds\,du\,dt\,dr\right\}^2, \tag{3.2.7}$$

where C_p is defined in (2.4.8) and C_0, K_p are as defined in (2.3.2).

Proof of Theorem 3.2.6 Note that, under H_0, taking $\underline{A}\,\Sigma\,\underline{A}^T = \underline{I}_{p-1}$, $\underline{Z}_1, \ldots, \underline{Z}_n$ are i.i.d. with density of the form

$$f_{\underline{Z}}(\underline{z}) = K_p g(\underline{z}^T\underline{z}) = K_p \int_{-\infty}^{\infty} \exp\{-[(\underline{z}^T\underline{z}+s^2)/C_0]^{\nu}\}ds,\ \underline{z} \in R^{p-1}. \tag{3.2.8}$$

(See Lemma 2.3.3.) Under the sequence of alternatives, described in section 2.3, $\underline{Z}_1, \ldots, \underline{Z}_n$ are i.i.d. with density $f_{\underline{Z}}(\underline{z} - \underline{c}/\sqrt{n}\,)$, where $\underline{c} \in R^{p-1}$-{0}. We have shown that $0 < I_{\underline{c}}(f_{\underline{Z}}) < \infty$ if $4\nu + p > 3$ (see Appendix A), thus, under the sequence of alternatives, it follows from Lemma 3.2.3 that

$$nW_n \xrightarrow{d} \chi^2_{p-1}\left(\frac{12}{p-1}\left\{E_{H_0}\left[\frac{K(R^2)Rg'(R^2)}{g(R^2)}\right]\right\}^2 \underline{c}^T\underline{c}\right)$$

where

$$g'(R^2) = dg(R^2)/dR^2 \ \text{ and } \ R^2 = \underline{Z}^T\underline{Z}.$$

Therefore,

$$\mathrm{ARE}(nW_n,\ T^2) = \frac{12}{p-1}\left\{E_{H_0}\left[\frac{K(R^2)Rg'(R^2)}{g(R^2)}\right]\right\}^2. \tag{3.2.9}$$

Recall Lemma 2.4.2 that $R^2 = \underline{Z}^T\underline{Z}$ has density of the form

$$f_{R^2}(r) = \frac{K_p \pi^{(p-1)/2}}{\Gamma[(p-1)/2]} r^{(p-1)/2-1} g(r),\ r > 0. \tag{3.2.10}$$

Thus, we have

$$E_{H_0}\left[\frac{K(R^2)Rg'(R^2)}{g(R^2)}\right]$$

$$= \int_0^\infty \left[\frac{K(r)\sqrt{r}\, g'(r)}{g(r)}\right] f_{R^2}(r)\, dr$$

$$= C_p \int_0^\infty \left[\frac{K(r)\sqrt{r}\, g'(r)}{g(r)}\right] r^{(p-1)/2-1} g(r)\, dr$$

$$= C_p \int_0^\infty K(r) g'(r) r^{p/2-1}\, dr.$$

Note that

$$g'(r) = \left(\frac{-2\nu}{C_0^\nu}\right) \int_0^\infty (r+t^2)^{\nu-1} \exp\{-[(r+t^2)/C_0]^\nu\}\, dt.$$

It follows that

$$E_{H_0}\left[\frac{K(R^2)Rg'(R^2)}{g(R^2)}\right]$$

$$= \left(\frac{-2\nu \cdot C_p}{C_0^\nu}\right) \int_0^\infty \int_0^\infty K(r) \cdot r^{p/2-1} (r+t^2)^{\nu-1} \exp\{-[(r+t^2)/C_0]^\nu\}\, dt\, dr$$

$$= \left(\frac{-2\nu \cdot C_p}{C_0^\nu}\right) \int_0^\infty \int_0^\infty \left[\int_0^r f_{R^2}(s)\, ds\right] r^{p/2-1} (r+t^2)^{\nu-1} \exp\{-[(r+t^2)/C_0]^\nu\}\, dt\, dr.$$

Using the expressions (3.2.10) and (3.2.8) for f_{R^2} and g, respectively, in the above expression yields

$$E_{H_0}\left[\frac{K(R^2)Rg'(R^2)}{g(R^2)}\right]$$

$$= \left(\frac{-4\nu\, C_p^2}{C_0^\nu}\right) \int_0^\infty\int_0^\infty\int_0^\infty\int_0^r s^{(p-1)/2-1}\, r^{p/2-1}\, (r+t^2)^{\nu-1}\ \exp\{-[(s+u^2)/C_0]^\nu\}\cdot$$

$$\exp\{-[(r+t^2)/C_0]^\nu\}ds\, du\, dt\, dr$$

$$= \left(\frac{-16\nu\, C_p^2}{C_0^\nu}\right) \int_0^\infty\int_0^\infty\int_0^\infty\int_0^r s^{p-2} r^{p-1}\, (r^2+t^2)^{\nu-1} \exp\{-[(s^2+u^2)/C_0]^\nu\}\cdot$$

$$\exp\{-[(r^2+t^2)/C_0]^\nu\}\ ds\, du\, dt\, dr.$$

Substituting the above expression into (3.2.9) yields Theorem 3.2.6.

3.3 Numerical Evaluation of ARE(nW_n, T^2)

In this section we describe the numerical evaluation of the ARE expression given in Theorem 3.2.6. All calculations were performed with an IBM computer running on a VM/CMS operating system using fortran 77. Simpson's rule and the IMSL subroutine DMLIN were used to integrate single integrals and 3-dimensional integrals, respectively.

Now we describe how the calculations were performed as well as the error associated with them. Letting

$$u^* = e^{-u},\ t^* = e^{-t},\ \text{and}\ r^* = e^{-r},$$

we can write

$$\sqrt{ARE(nW_n, T^2)}$$

$$= \sqrt{\frac{12}{p-1}}\left(\frac{16\nu\, C_p^2}{C_0^{\nu}}\right)\Big\{ \int_0^1\int_0^1\int_0^1\int_0^{-\ell n r^*} s^{p-2}(-\ell n r^*)^{p-1}\,(r^* t^* u^*)^{-1}[(\ell n t^*)^2+(\ell n r^*)^2]^{\nu-1}\cdot$$

$$\exp\{-[(s^2+(\ell n u^*)^2)/C_0]^{\nu}\}\exp\{-[((\ell n r^*)^2+(\ell n t^*)^2)/C_0]^{\nu}\}ds\, du^* dt^* dr^*\Big\}$$

$$= \int_0^1\int_0^1\int_0^1\Big[\int_0^{-\ell n r} 4C_p s^{p-2}u^{-1}\exp\{-[(s^2+(\ell n u)^2)/C_0]^{\nu}\}\ ds\ \Big]\cdot$$

$$\Big[\sqrt{\frac{12}{p-1}}\frac{4\nu C_p}{C_0^{\nu}}(-\ell n r)^{p-1}[(\ell n t)^2+(\ell n r)^2]^{\nu-1}(t\, r)^{-1}\exp\{-[((\ell n r)^2+(\ell n t)^2)/C_0]^{\nu}\}\Big]du\, dt\, dr$$

$$= \int_0^1\int_0^1\int_0^1\Big[\int_0^{-\ell n r} k(u, s)\, ds\Big]\, h(t, r)\, du\, dt\, dr, \tag{3.3.1}$$

where

$$k(u, s) = 4C_p s^{p-2}u^{-1}\exp\{-[(s^2+(\ell n u)^2)/C_0]^{\nu}\}\ ,$$

$$h(t, r) = \sqrt{\frac{12}{p-1}}\ \frac{4\nu\, C_p}{C_0^{\nu}}(-\ell n r)^{p-1}[(\ell n t)^2+(\ell n r)^2]^{\nu-1}(t\, r)^{-1}\exp\{-[((\ell n r)^2+(\ell n t)^2)/C_0]^{\nu}\},$$

and ℓn is the natural logarithm. Note that, following standard arguments and using expressions (2.4.7) and (2.4.9),

$$\int_0^1\int_0^1 h(t, r)\, dt\, dr = \sqrt{3ARE(V_n, T^2)}\ , \tag{3.3.2}$$

where $ARE(V_n, T^2)$ is given in Theorem 2.4.5. Using Simpson's rule with an absolute error $\varepsilon_1/\sqrt{3ARE(V_n, T^2)}$, we can approximate the inner single integral in expression (3.3.1) by $g(u, r)$ satisfying

$$\Big|\ \int_0^{-\ell n r} k(u, s)\, ds - g(u, r)\ \Big| \le \varepsilon_1/\sqrt{3ARE(V_n, T^2)}. \tag{3.3.3}$$

Hence, using expressions (3.3.2) and (3.3.3), we can write expression (3.3.1) as

$$\sqrt{\mathrm{ARE}(nW_n, T^2)}$$

$$\approx \int_0^1\int_0^1\int_0^1 \left[\, g(u, r) \pm \varepsilon_1/\sqrt{3\mathrm{ARE}(V_n, T^2)}\,\right] h(t, r)\, du\, dt\, dr$$

$$= \int_0^1\int_0^1\int_0^1 g(u, r)h(t, r)\, du\, dt\, dr \pm \left[\varepsilon_1/\sqrt{3\mathrm{ARE}(V_n, T^2)}\,\right] \int_0^1\int_0^1 h(t, r)\, dt\, dr$$

$$= \int_0^1\int_0^1\int_0^1 g(u, r)h(t, r)\, du\, dt\, dr \pm \varepsilon_1. \qquad (3.3.4)$$

Now, using the IMSL subroutine DMLIN with an absolute error ε_2, we can approximate the above 3-dimensional integral by VAL satisfying

$$\left| \int_0^1\int_0^1\int_0^1 g(u, r)\cdot h(t, r)\, du\, dt\, dr - \mathrm{VAL} \right| \le \varepsilon_2. \qquad (3.3.5)$$

Combining expressions (3.3.4) and (3.3.5), we have

$$\sqrt{\mathrm{ARE}(nW_n, T^2)} \approx \mathrm{VAL} \pm (\varepsilon_1+\varepsilon_2),$$

where VAL satisfies expression (3.3.5). Thus, we approximate $\mathrm{ARE}(nW_n, T^2)$ by $(\mathrm{VAL})^2$ with the maximum error

$$\mathrm{ERREST} = 2\mathrm{VAL}(\varepsilon_1+\varepsilon_2) + (\varepsilon_1+\varepsilon_2)^2. \qquad (3.3.6)$$

In Table 3.1 we present the asymptotic relative efficiencies for nW_n with respect to T^2 for selected values of ν and p satisfying the condition $4\nu+p > 3$. We also include $\mathrm{ARE}(V_n, T^2)$, in parentheses, for easy comparison. When the underlying population is multivariate normal ($\nu = 1.0$), Hotelling's T^2 performs well. Both the sign test and signed-rank test appear to be quite competitive, with the signed-rank test slightly better than the sign test provided the dimension is not too large. For light-tailed distributions

(ν = 2.0 and 5.0) and p > 2, the signed-rank test W_n has greater power compared to both Hotelling's T^2 and V_n. For heavy-tailed distributions (ν = .50 and .10), V_n is clearly most powerful, although, W_n still performs well relative to Hotelling's T^2, provided the dimension is not too large.

In Table 3.2 we display the values of ERREST, defined in expression (3.3.6), which bound the error in the estimates of ARE(nW_n, T^2). Assuming $\varepsilon_1 = \varepsilon_2 = \varepsilon$, where ε_1 and ε_2 are defined in (3.3.3) and (3.3.5), respectively, we use the folling ε values : ε = .001 when ν = 1.0, ε = .001 when ν = 2.0 and p =2 to 6, ε = .005 when ν = 2.0 and p =7 or 8, ε = .01 when ν = 5.0 and p = 2 to 6, ε = .02 when ν = 5.0 and p = 7 or 8, ε = .001 when ν = .50, ε = .10 when ν = .10 and p = 2, and ε = .015 when ν = .10 and p = 3 to 8.

Table 3.1

ARE(nW_n, T^2), with ARE(V_n, T^2) in parentheses.

p	ν = 5.0	2.0	1.0	.50	.10
2	.8674 (.4205)	.8779 (.4784)	.9550 (.6366)	1.2658 (1.2159)	36.4044* (297.1379*)
3	1.0834 (.5738)	1.0124 (.6364)	.9855 (.7854)	1.0786 (1.2337)	8.4010 (50.7101)
4	1.1809 (.6599)	1.0535 (.7191)	.9748 (.8488)	.9873 (1.2008)	4.1877 (20.1730)
5	1.2208 (.7164)	1.0664 (.7708)	.9614 (.8836)	.9347 (1.1711)	2.7726 (11.4238)
6	1.2476 (.7566)	1.0683 (.8063)	.9491 (.9054)	.9012 (1.1477)	2.0814 (7.7606)
7	1.2525 (.7868)	1.0593 (.8323)	.9382 (.9204)	.8777 (1.1295)	1.7042 (5.8638)
8	1.2574 (.8103)	1.0592 (.8521)	.9289 (.9313)	.8605 (1.1151)	1.4895 (4.7408)

*Conjectured ARE's when $p = 2$ and $\nu \leq .25$, using the expressions in Theorem 3.2.6 and Theorem 2.4.5 for ARE(nW_n, T^2) and ARE(V_n, T^2), respectively.

Table 3.2

Error Estimate ERREST of $ARE(nW_n, T^2)$

	ν				
p	5.0	2.0	1.0	.50	.10
2	.0377	.0038	.0039	.0045	2.4534
3	.0420	.0040	.0040	.0042	.1748
4	.0439	.0041	.0040	.0040	.1237
5	.0446	.0041	.0039	.0039	.1008
6	.0451	.0041	.0039	.0038	.0875
7	.0911	.0207	.0039	.0038	.0792
8	.0913	.0207	.0039	.0037	.0741

CHAPTER 4
MONTE CARLO STUDY

In this chapter we display results from a Monte Carol study when the dimension is $p = 4$, the sample size is $n = 20$, and the significance level is $\alpha = .05$. In addition to the affine-invariant statistics T^2, V_n, and nW_n, we examined two other nonaffine-invariant statistics. The ANOVA F test is included along with the rank transformation test RT, which were both introduced in chapter 1. These five test statistics were compared under five different distributions. They were quadrivariate normal distribution, elliptically symmetric distributions with density of the form given in (2.3.1), Pearson Type II and Type VII (see Johnson [1987], p.111 and p.117-118, respectively), and the mixtures of quadrivariate normal distributions. (A quadrivariate normal mixture is obtained by selecting randomly one of two quadrivariate normal distributions. Each of the observations is sampled with probability p from the first distribution and with probability 1-p from the second distribution. See Johnson [1987], p.56-57.) These distributions were located at $\underline{\mu}_a = (m\delta, m\delta, m\delta, 0)$, $m = 0, 1, 2, 3$, for the original sample $\underline{Y}_1, \ldots, \underline{Y}_n$, on which ANOVA F test and rank transformation test RT were applied, and they were located at $\underline{\theta}_a = (m\delta, m\delta, m\delta)$, $m = 0, 1, 2, 3$, for the transformed sample $\underline{Z}_1, \ldots, \underline{Z}_n$, on which the tests T^2, V_n, and nW_n were performed. The value of δ was adjusted for different distributions to examine somewhat silimar points on the power curves. Since the performances of tests ANOVA F and RT depend on the variance-covariance structure of the distribution, for each of the above first four distributions, we considered two types of variance-covariance matrices, one with $\underline{I}_4$, the identity matrix, and the other one with a non H-type structure. For the mixtures of normal distributions, we consider one mixture with a non H-type structure, and three other mixtures with H-type structures. In each Monte Carlo simulation,

the proportion of times out of 1000 in which each test statistic exceeded the upper α-percentile of its null distribution is reported. The asymptotic null distribution χ_3^2 is used to determine the critical value for tests V_n and nW_n. For tests ANOVA F and rank transformation RT, the null distribution $F_{3,57}$ is used to determine the critical value. While for Hotelling's T^2, we use the null distribution $\frac{57}{17}F_{3,17}$ (namely, a multiple of $F_{3,17}$) to determine the critical value. All calculations and random variables generations were performed with an IBM computer running on a VM/CMS operating system using fortran 77. Several IMSL subroutines, to be described later in this chapter, were used.

In Tables 4.1 and 4.2, the results from the Monte Carol studies for quadrivariate normal distributions with $\underline{\Sigma} = \underline{I}_4$ and $\underline{\Sigma} = \underline{E}\,\underline{E}^T$, respectively, are presented. Note that we used

$$\underline{E} = \begin{bmatrix} 1 & 3 & 0 & 2 \\ -2 & 1 & 4 & -1 \\ 1 & -1 & 1 & 0 \\ -2 & 4 & -1 & 3 \end{bmatrix}, \tag{4.1.1}$$

producing

$$\underline{E}\,\underline{E}^T = \begin{bmatrix} 14 & -1 & -2 & 16 \\ -1 & 22 & 1 & 1 \\ -2 & 1 & 3 & -7 \\ 16 & 1 & -7 & 30 \end{bmatrix} \tag{4.1.2}$$

which is non H-type. The IMSL subroutine GGNML was used to generate N(0, 1) variables. As indicated, 1000 samples of size 20 and significance level .05 were used. When $\underline{\Sigma} = \underline{I}_4$, as we may expect, ANOVA F and rank transformation test RT have better power than Hotelling's T^2. And they all perform better than V_n and nW_n (see Table 4.1). However, when $\underline{\Sigma} = \underline{E}\,\underline{E}^T$, Hotelling's T^2 has good power, followed by V_n and nW_n, both performing better than F and RT (see Table 4.2). This illustrates the strong dependence of the performances of tests F and RT on the variance-covariance structure of the distribution. For both variance-covariance structures, ANOVA F performs better than RT, and nW_n

performs slightly better than V_n, which agrees with the result found in Table 3.1 for $\nu = 1.0$ and $p = 4$.

In Tables 4.3 and 4.4, we display the Monte Carlo results for five members of the class of elliptically symmetric distributions with density of the form given in (2.3.1) with $\underline{\Sigma} = \underline{I}_4$ and $\underline{\Sigma} = \underline{E}\,\underline{E}^T$, respectively. Heavy-tailed distributions, $\nu = .10$ and $.50$, the quadrivariate normal distribution, $\nu = 1.0$, (repeated in this table to permit comparisons), and light-tailed distributions, $\nu = 50.0$ and 100.0, were included. Taking $\underline{\mu} = \underline{0}$ and $\underline{\Sigma} = \underline{I}_4$ in (2.3.1), it follows from standard arguments that

$$P(R^2 \le w) = P(C_0 G^{1/\nu} \le w),\ w \ge 0,$$

where $R^2 = \underline{Y}^T\,\underline{Y}$, C_0 is defined in (2.3.2), and G has the distribution of Gamma(1, p/2ν). Thus $\underline{Y}$ can be generated via $\underline{Y} = R\,\underline{U}$, where R is independent of $\underline{U}$ having the distribution of $\sqrt{C_0}G^{1/2\nu}$, and $\underline{U}$ is uniformly distributed on the unit-sphere S_p (see Johnson [1987], p.110). Here, the IMSL subroutines GGAMR and GGSPH were used to generate Gamma(1, r) and the variables that are uniformly distributed on the unit-sphere S_p, respectively. Although it is difficult to compare powers for statistics with different rejection proportions under the null hypothesis, it appears that the Monte Carol results tend to agree with the asymptotic results of the previous two chapters. For light-tailed distributions (ν = 50.0 and 100.0), ANOVA F and RT have great power if $\underline{\Sigma} = \underline{I}_4$, while Hotelling's T^2 and nW_n have the greatest power when $\underline{\Sigma} = \underline{E}\,\underline{E}^T$. For the heaviest-tailed distribution ($\nu = .10$) examined, V_n performs best, followed in order by RT and Hotelling's T^2 when $\underline{\Sigma} = \underline{I}_4$, and by Hotelling's T^2 and RT when $\underline{\Sigma} = \underline{E}\,\underline{E}^T$. Generally speaking, for light-tailed distributions (ν = 50.0 and 100.0), the signed-rank test nW_n performs better than the sign test V_n, and ANOVA F performs better than RT; however, for heavy-tailed distributions (ν = .10 and .50), the interdirection sign test V_n performs better than nW_n, and RT works better than F. The superiority of tests ANOVA F and nW_n (RT and V_n) for very light-tailed (heavy-tailed) distributions is shown here. Except for the heaviest-tailed distribution ($\nu = .10$), ANOVA F

and RT both perform better than V_n and nW_n when $\underline{\Sigma} = \underline{I}_4$, however, V_n and nW_n both perform better than ANOVA F and RT when $\underline{\Sigma} = \underline{E}\,\underline{E}^T$. Also, we notice that RT performs better than Hotelling's T^2 only when $\underline{\Sigma} = \underline{I}_4$.

Next, we consider the light-tailed Pearson Type II distributions, a special case of elliptically symmetric distributions, with the density function

$$f_{\underline{Y}}(\underline{y}) = \frac{\Gamma(m+3)}{\Gamma(m+1)\pi^2} |\underline{\Sigma}|^{-1/2}\{1-(\underline{y}-\underline{\mu})^T\underline{\Sigma}^{-1}(\underline{y}-\underline{\mu})\}^m, \tag{4.1.3}$$

having support $(\underline{y}-\underline{\mu})^T\underline{\Sigma}^{-1}(\underline{y}-\underline{\mu}) \leq 1$ and shape parameter $m > -1$. The Monte Carlo results for this family of distributions with $\underline{\Sigma} = \underline{I}_4$ and $\underline{\Sigma} = \underline{E}\,\underline{E}^T$ are presented in Tables 4.5 and 4.6, respectively. Taking $\underline{\mu} = \underline{0}$ and $\underline{\Sigma} = \underline{I}_4$ in (4.1.3), it can be shown that $R^2 = \underline{Y}^T\,\underline{Y}$ has the distribution of Beta(2, m+1). Thus, the variable generation is similar to that in the last case considered. (See, Johnson [1987], p.116.) Here, the IMSL subroutine GGBTR was used to generate beta variables. An examination of Tables 4.5 and 4.6 indicates that ANOVA F and RT have great power if $\underline{\Sigma} = \underline{I}_4$, while Hotelling's T^2 and nW_n have the greatest power when $\underline{\Sigma} = \underline{E}\,\underline{E}^T$. The signed-rank test nW_n performs better than the sign test V_n, and ANOVA F performs better than RT. Furthermore, ANOVA F and RT both perform better than V_n and nW_n when $\underline{\Sigma} = \underline{I}_4$, while V_n and nW_n both perform better than ANOVA F and RT when $\underline{\Sigma} = \underline{E}\,\underline{E}^T$. These results agree with the findings in Tables 4.3 and 4.4 for the light-tailed elliptically symmetric distributions.

For the heavy-tailed Pearson Type VII distributions, the Monte Carlo results are presented in Tables 4.7 and 4.8 for $\underline{\Sigma} = \underline{I}_4$ and $\underline{\Sigma} = \underline{E}\,\underline{E}^T$, respectively. The density function has the form of

$$f_{\underline{Y}}(\underline{y}) = \frac{\Gamma(m)}{\Gamma(m-2)\pi^2} |\underline{\Sigma}|^{-1/2}\{1+(\underline{y}-\underline{\mu})^T\underline{\Sigma}^{-1}(\underline{y}-\underline{\mu})\}^{-m}, \tag{4.1.4}$$

where the shape parameter $m > 2$. Note that, taking $\underline{\mu} = \underline{0}$ and $\underline{\Sigma} = \underline{I}_4$ in (4.1.4), $\underline{Y}$ can be generated via $\underline{Y} = \underline{Z}/\sqrt{S}$ (see Johnson [1987], p. 118), where $\underline{Z}$ is quadrivariate normal

with mean $\underline{0}$ and variance-covariance matrix $\underline{I}_4$, and S is independent of $\underline{Z}$ having the distribution of χ^2_b, b = 2m – 4. Note that when m = 2.5, $\underline{Y}$ has a multivariate cauchy distribution. In Tables 4.7 and 4.8, the sign test V_n has the greatest power. Notice that V_n performs better than nW_n, and RT works better than ANOVA F. Also, V_n and nW_n both perform better than F and RT when $\underline{\Sigma} = \underline{E}\,\underline{E}^T$. Note that the same general trends as in the heavy-tailed elliptically symmetric distributions are seen here.

Finally in Table 4.9 we examine samples from quadrivariate normal mixtures violating the assumption of elliptical symmetry required for the asymptotic results of the previous two chapters. As we may expect, the sign test V_n and signed-rank test nW_n appear to be quite robust. For each mixture in Table 4.9, the parameters of the distributions ($\underline{\mu}_1$, $\underline{\mu}_2$, $\underline{\Sigma}_1$, and $\underline{\Sigma}_2$) and the mixing probability (p) as well as the parameters of the mixture ($\underline{\mu}$ and $\underline{\Sigma}$), computed by

$$\underline{\mu} = p\underline{\mu}_1 + (1\text{-}p)\underline{\mu}_2,$$

and

$$\underline{\Sigma} = p\underline{\Sigma}_1 + (1\text{-}p)\underline{\Sigma}_2 + p(1\text{-}p)(\underline{\mu}_1 - \underline{\mu}_2)(\underline{\mu}_1 - \underline{\mu}_2)^T, \qquad (4.1.5)$$

(see, Johnson [1987], p.57), are indicated. In mixture 1, a light-tailed distribution with H-type structure, we see that tests ANOVA F, RT, and Hotelling's T^2 have great power, and all performing better than V_n and nW_n. This agrees with that of Table 4.1, for normal distribution with $\underline{\Sigma} = \underline{I}_4$. An examination of the table for mixtures 2 and 3, heavy-tailed distributions with H-type structure, shows the same general trends. That is, the superiority of tests V_n and RT over the signed-rank test nW_n and the other competitors are seen. Also, by comparing with the result for elliptically symmetric distribution with $\nu = .10$ and $\underline{\Sigma} = \underline{I}_4$ in Table 4.3, and that for Pearson Type VII distribution with m = 3.0 and $\underline{\Sigma} = \underline{I}_4$ in Table 4.7, it illustrates that the performances of Hotelling's T^2 and ANOVA F are affected by the

asymmetry of the distributions, which as a result makes nW_n a better test than Hotelling's T^2 and ANOVA F. Finally, in mixture 4, a heavy-tailed distribution with non H-type structure, the robustness and superiority of tests V_n and nW_n is easily seen.

In summary, for normal to light-tailed symmetric distributions and a variance-covariance structure that is non H-type, Hotelling's T^2 has the greatest power, followed by the signed-rank test nW_n. When the structure is H-type and the populations are normal or light-tailed the best is ANOVA F, followed by RT. For the heavy-tailed distributions with H-type variance-covariance structure, the interdirection sign test V_n has the greatest power, followed by RT. For heavy-tailed symmetric distributions with non H-type variance-covariance structure, the interdirection sign test V_n still has the greatest power, but this time Hotelling's T^2 is second best. For the asymmetric distribution with non H-type variance-covariance structure, the interdirection sign test V_n and the signed-rank test nW_n are the best. Generally, the tests V_n and nW_n both perform better than ANOVA F and RT when the variance-covariance structure is non H-type while the contrary is true only for normal to light-tailed distributions with H-type variance-covariance structures. This suggests that V_n and nW_n are promising procedures for the repeated measures settings. The tests ANOVA F and RT follow the same general patterns as nW_n and V_n, respectively. That is, the RT (V_n) test performs better than ANOVA F (nW_n) for heavy-tailed distributions, and ANOVA F (nW_n) performs better than RT (V_n) for light-tailed distributions. Also note that test RT works better than Hotelling's T^2 for distributions with H-type variance-covariance structures. Finally, we should point out that the performance of the test based on nW_n is disappointing, since the power of nW_n is not always better than that of Hotelling's T^2 for light-tailed symmetric distributions (see Tables 4.3 and 4.4). One explaination to this may be because of the higher probability of Type I error of Hotelling's T^2 test.

Table 4.1

Monte Carlo Results for Quadrivariate Normal Distribution with $\underline{\Sigma} = \underline{I}_4$.

Amount of Shift	Statistics				
	F	RT	T^2	V_n	nW_n
.00	.049	.046	.056	.052	.058
.25	.106	.113	.108	.082	.108
.50	.335	.321	.300	.219	.247
.75	.649	.644	.579	.457	.485

Entries : Proportion of times each test statistic exceeded the upper α-percentile of its null distribution.

Dimension : $p = 4$.

Sample Size : $n = 20$.

Number of Samples : rep = 1000.

Significance Level : $\alpha = 0.5$.

Table 4.2

Monte Carlo Results for Quadrivariate Normal Distribution with $\underline{\Sigma} = \underline{E}\,\underline{E}^T$.

Amount of Shift	Statistics				
	F	RT	T^2	V_n	nW_n
.00	.081	.092	.055	.040	.056
.60	.091	.101	.108	.088	.114
1.20	.164	.144	.283	.204	.235
1.80	.279	.248	.540	.430	.453

Entries : Proportion of times each test statistic exceeded the upper α-percentile of its null distribution.

Dimension : p = 4.

Sample Size : n = 20.

Number of Samples : rep = 1000.

Significance Level : $\alpha = 0.5$.

Table 4.3

Monte Carlo Results for Elliptically Symmetric Distributions with Density of the Form Given in (2.3.1) with $\underline{\Sigma} = \underline{I}_4$.

Amount of Shift	Statistics				
	F	RT	T^2	V_n	nW_n
	$\nu = .10$				
.00	.026	.048	.016	.032	.040
.17	.079	.226	.097	.276	.124
.34	.248	.636	.367	.669	.288
.51	.500	.872	.626	.894	.432
	$\nu = .50$				
.00	.053	.057	.050	.032	.048
.25	.110	.119	.101	.079	.088
.50	.321	.368	.303	.260	.234
.75	.684	.699	.618	.573	.462
	$\nu = 1.0$				
.00	.049	.046	.056	.052	.058
.25	.106	.113	.108	.082	.108
.50	.335	.321	.300	.219	.247
.75	.649	.644	.579	.457	.485

Table 4.3 - - continued.

Amount of Shift	Statistics				
	F	RT	T^2	V_n	nW_n
	$\nu = 50.0$				
.00	.046	.049	.059	.032	.044
.25	.100	.085	.092	.057	.087
.50	.316	.279	.247	.166	.261
.75	.656	.594	.559	.361	.568
	$\nu = 100.0$				
.00	.050	.053	.053	.032	.048
.25	.095	.089	.091	.056	.088
.50	.317	.282	.256	.159	.262
.75	.650	.584	.573	.354	.557

Entries : Proportion of times each test statistic exceeded the upper α-percentile of its null distribution.

Dimension : $p = 4$.

Sample Size : $n = 20$.

Number of Samples : rep = 1000.

Significance Level : $\alpha = 0.5$.

Table 4.4

Monte Carlo Results for Elliptically Symmetric Distributions with Density of the Form Given in (2.3.1) with $\underline{\Sigma} = \underline{E}\,\underline{E}^T$.

Amount of Shift	Statistics				
	F	RT	T^2	V_n	nW_n
	$\nu = .10$				
.00	.055	.081	.021	.049	.037
.40	.073	.128	.094	.277	.127
.80	.124	.277	.325	.657	.268
1.20	.224	.462	.586	.879	.418
	$\nu = .50$				
.00	.079	.081	.060	.049	.063
.60	.094	.104	.110	.100	.107
1.20	.154	.152	.300	.266	.238
1.80	.268	.219	.609	.558	.452
	$\nu = 1.0$				
.00	.081	.092	.055	.040	.056
.60	.091	.101	.108	.088	.114
1.20	.164	.144	.283	.204	.235
1.80	.279	.248	.540	.430	.453

Table 4.4 - - continued.

Amount of Shift	Statistics				
	F	RT	T^2	V_n	nW_n
	$\nu = 50.0$				
.00	.078	.079	.060	.049	.048
.80	.110	.105	.139	.098	.124
1.60	.197	.163	.438	.270	.434
2.40	.390	.275	.833	.620	.830
	$\nu = 100.0$				
.00	.073	.080	.065	.049	.055
.80	.106	.109	.138	.099	.123
1.60	.203	.157	.428	.275	.447
2.40	.394	.280	.824	.617	.822

Entries : Proportion of times each test statistic exceeded the upper α-percentile of its null distribution.

Dimension : $p = 4$.

Sample Size : $n = 20$.

Number of Samples : rep = 1000.

Significance Level : $\alpha = 0.5$.

Table 4.5

Monte Carlo Results for Pearson Type II Distribution with Shape Parameter m, and Density of the Form Given in (4.1.3) with $\underline{\Sigma} = \underline{I}_4$.

Amount of Shift	Statistics				
	F	RT	T^2	V_n	nW_n
	m = 1.0				
.00	.045	.045	.050	.032	.039
.10	.115	.106	.108	.064	.092
.20	.402	.345	.340	.221	.292
.30	.771	.724	.687	.493	.649
	m = 2.0				
.00	.052	.045	.051	.032	.053
.10	.136	.130	.122	.077	.113
.20	.498	.451	.415	.281	.370
.30	.867	.830	.788	.601	.726

Entries : Proportion of times each test statistic exceeded the upper α-percentile of its null distribution.

Dimension : $p = 4$.

Sample Size : $n = 20$.

Number of Samples : rep = 1000.

Significance Level : $\alpha = 0.5$.

Table 4.6

Monte Carlo Results for Pearson Type II Distribution with Shape Parameter m, and Density of the Form Given in (4.1.3) with $\underline{\Sigma} = \underline{E}\,\underline{E}^T$.

Amount of Shift	Statistics				
	F	RT	T^2	V_n	nW_n
	m = 1.0				
.00	.078	.080	.060	.049	.053
.30	.111	.109	.153	.111	.144
.60	.223	.170	.490	.325	.454
.90	.438	.330	.877	.704	.832
	m = 2.0				
.00	.072	.075	.059	.049	.054
.30	.118	.113	.172	.125	.160
.60	.275	.195	.587	.425	.522
.90	.540	.403	.930	.812	.885

Entries : Proportion of times each test statistic exceeded the upper α-percentile of its null distribution.

Dimension : p = 4.

Sample Size : n = 20.

Number of Samples : rep = 1000.

Significance Level : $\alpha = 0.5$.

Table 4.7

Monte Carlo Results for Pearson Type VII Distribution with Shape Parameter m, and Density of the Form Given in (4.1.4) with $\underline{\Sigma} = \underline{I}_4$.

Amount of Shift	Statistics				
	F	RT	T^2	V_n	nW_n
	m = 2.5				
.00	.019	.052	.022	.052	.050
.40	.024	.106	.042	.107	.097
.80	.067	.262	.109	.287	.190
1.20	.120	.476	.214	.520	.312
	m = 3.0				
.00	.035	.046	.030	.052	.058
.50	.179	.324	.226	.295	.220
1.00	.592	.863	.661	.829	.575
1.50	.851	.985	.901	.976	.777

Entries : Proportion of times each test statistic exceeded the upper α-percentile of its null distribution.

Dimension : p = 4.

Sample Size : n = 20.

Number of Samples : rep = 1000.

Significance Level : $\alpha = 0.5$.

Table 4.8

Monte Carlo Results for Pearson Type VII Distribution with Shape Parameter m, and Density of the Form Given in (4.1.4) with $\underline{\Sigma} = \underline{E}\,\underline{E}^T$.

Amount of Shift	Statistics				
	F	RT	T^2	V_n	nW_n
	m = 2.5				
.00	.025	.077	.027	.040	.048
1.20	.031	.098	.058	.135	.110
2.40	.054	.173	.149	.376	.232
3.60	.087	.294	.281	.653	.375
	m = 3.0				
.00	.059	.080	.027	.040	.051
1.10	.093	.137	.182	.251	.199
2.20	.228	.340	.567	.709	.508
3.30	.442	.620	.848	.939	.715

Entries : Proportion of times each test statistic exceeded the upper α-percentile of its null distribution.

Dimension : p = 4.

Sample Size : n = 20.

Number of Samples : rep = 1000.

Significance Level : $\alpha = 0.5$.

Table 4.9

Monte Carlo Results for Quadrivariate Normal Mixtures with Mean $\underline{\mu}$ and Variance-Covariance Matrix $\underline{\Sigma}$ given in (4.1.5).

Amount of Shift	Statistics				
	F	RT	T^2	V_n	nW_n
	Mixture 1 $p = .5,\ \underline{\mu}_1 = [1, 1, 1, 1]^T = -\underline{\mu}_2,\ \underline{\Sigma}_1 = \underline{\Sigma}_2 = \underline{I}_4,$ $\underline{\mu} = \underline{0},\ \underline{\Sigma} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$				
.00	.049	.051	.047	.033	.049
.35	.190	.179	.171	.121	.160
.70	.604	.565	.528	.405	.428
1.05	.929	.915	.896	.800	.770
	Mixture 2 $p = .9,\ \underline{\mu}_1 = \underline{\mu}_2 = \underline{0},\ \underline{\Sigma}_1 = \underline{I}_4,\ \underline{\Sigma}_2 = 400 \cdot \underline{I}_4,$ $\underline{\mu} = \underline{0},\ \underline{\Sigma} = 40.9 \cdot \underline{I}_4.$				
.00	.021	.050	.031	.046	.050
.35	.037	.151	.078	.102	.125
.70	.096	.427	.231	.336	.317
1.05	.166	.744	.423	.668	.589

Table 4.9 - - continued.

Amount of Shift	Statistics				
	F	RT	T^2	V_n	nW_n
	Mixture 3 $p = .5,\ \ \underline{\mu}_1 = [1, 1, 1, 1]^T = -\underline{\mu}_2,\ \ \underline{\Sigma}_1 = \underline{I}_4,\ \ \underline{\Sigma}_2 = 400 \cdot \underline{I}_4,$ $\underline{\mu} = \underline{0},\ \ \underline{\Sigma} = \begin{bmatrix} 201.5 & 1 & 1 & 1 \\ 1 & 201.5 & 1 & 1 \\ 1 & 1 & 201.5 & 1 \\ 1 & 1 & 1 & 201.5 \end{bmatrix}.$				
.00	.035	.050	.024	.033	.038
1.40	.055	.229	.047	.332	.149
2.80	.090	.444	.088	.654	.216
4.20	.172	.543	.175	.769	.212
	Mixture 4 $p = .9,\ \ \underline{\mu}_1 = \underline{\mu}_2 = \underline{0},\ \ \underline{\Sigma}_1 = \underline{E}\,\underline{E}^T, \underline{\Sigma}_2 = 100 \cdot \underline{E}\,\underline{E}^T,$ $\underline{\mu} = \underline{0},\ \ \underline{\Sigma} = 10.9 \cdot \underline{E}\,\underline{E}^T$, $\underline{E}\,\underline{E}^T$ is defined in (4.1.2).				
.00	.033	.072	.030	.035	.053
1.60	.076	.144	.219	.292	.295
3.20	.224	.402	.617	.834	.723
4.80	.395	.738	.808	.986	.880

Entries : Proportion of times each test statistic exceeded the upper α-percentile of its null distribution.

Dimension : $p = 4$.

Sample Size : $n = 20$.

Number of Samples : rep = 1000.

Table 4.9 - - continued.

Significance Level : $\alpha = 0.5$.

Distributions : Quadrivarite normal mixture, choosing N_1 with probability p and N_2 with probability 1-p, where N_i, i =1, 2, is quadrivariate normal with mean $\underline{\mu}_i$ and variance-covariance matrix $\underline{\Sigma}_i$. The resulting mixture has mean $\underline{\mu}$ and variance-covariance matrix $\underline{\Sigma}$ (see (4.1.5)).

CHAPTER 5
A MULTIVARIATE SIGNED SUM TEST BASED ON INTERDIRECTIONS FOR THE ONE-SAMPLE LOCATION PROBLEM

5.1 Definition of the Test Statistic

We now leave the repeated measures problem, and consider the one-sample multivariate location problem. In this section we propose a multivariate signed sum test based on interdirections, described in section 2.1, for the one-sample multivariate location problem. To do so, we let $\underline{X}_1, \ldots, \underline{X}_n$, where $\underline{X}_i = (X_{i1}, \ldots, X_{ip})^T$, be i.i.d. as $\underline{X} = (X_1, \ldots, X_p)^T$, where $\underline{X}$ is from a p-variate absolutely continuous population with location parameter $\underline{\theta}^*$ (p×1). We would like to test

$$H_0 : \underline{\theta}^* = \underline{0} \quad \text{versus} \quad H_a: \underline{\theta}^* \neq \underline{0}. \tag{5.1.1}$$

Here $\underline{0}$ is used without loss of generality, since $H_0 : \underline{\theta}^* = \underline{\theta}_0$ can be tested by subtracting $\underline{\theta}_0$ from each observation $\underline{X}_i$, and testing whether the differences $(\underline{X}_i - \underline{\theta}_0)$'s are located at $\underline{0}$.

The procedure is somewhat like applying the multivariate sign test, proposed by Randles (1989) for the one-sample location problem, to the sums

$$\underline{X}_s + \underline{X}_t,\ 1 \leq s \leq t \leq n, \tag{5.1.2}$$

and rejects H_0 for large values of the statistic

$$SS = \frac{4p}{n(n+1)^2} \sum_{i \leq k} \sum_{i' \leq k'} \cos(\pi\, \hat{p}_{ik,i'k'}), \tag{5.1.3}$$

where

$$\widehat{p}_{ik,i'k'} = \begin{cases} \dfrac{C_{ik,i'k'}+d_n}{\binom{n}{p-1}} & \text{if } (i, k) \neq (i', k') \\[2ex] 0 & \text{if } (i, k) = (i', k'), \end{cases} \tag{5.1.4}$$

and $C_{ik,i'k'}$ denotes the number of hyperplanes formed by the origin 0 and p-1 of the other observations $\underline{X}_j$ (excluding $\underline{X}_i$, $\underline{X}_k$, $\underline{X}_{i'}$, and $\underline{X}_{k'}$) such that $\underline{X}_i + \underline{X}_k$ and $\underline{X}_{i'} + \underline{X}_{k'}$ are on opposite sides of the hyperplane formed. The counts $\{C_{ik,i'k'} \mid 1 \leq i \leq k \leq n,\ 1 \leq i' \leq k' \leq n\}$, called interdirections, are used via $\pi\widehat{p}_{ik,i'k'}$ to measure the angular distance between $\underline{X}_i + \underline{X}_k$ and $\underline{X}_{i'} + \underline{X}_{k'}$ relative to the positions of the other observations. Here $\widehat{p}_{ik,i'k'}$ is the observed fraction of times that $\underline{X}_i + \underline{X}_k$ and $\underline{X}_{i'} + \underline{X}_{k'}$ fall on opposite sides of the hyperplanes formed by $\underline{0}$ and other p-1 observations.

We now examine some characteristics of the test based on SS. First of all, it can be easily seen that the test SS defined in (5.1.3) is like the multivariate sign test V_n applied to the sums $\underline{X}_s + \underline{X}_t$, $1 \leq s \leq t \leq n$. Secondly, since the $\widehat{p}_{ik,i'k'}$, $1 \leq i \leq k \leq n$, $1 \leq i' \leq k' \leq n$, are invariant with respect to the nonsinglar linear transformations, as shown by Randles (1989), it is clear that SS is also affine-invariant. Thirdly, for $p > 1$, SS does not have a small sample distribution-free property. This is because the joint distribution of the directions of $\underline{X}_i + \underline{X}_k$, $1 \leq i \leq k \leq n$, from $\underline{0}$ depends on the distribution of the distances of $\underline{X}_i$, $1 \leq i \leq n$, from the origin. However, its large sample null distribution is convenient, as is shown in the next section. We end this section by proving a Lemma which shows that for $p = 1$, the test based on SS is the two-sided univariate Wilcoxon signed-rank test. Hence for $p = 1$ it does have a small sample distribution-free property, as well as a convenient large sample null distribution.

<u>Lemma 5.1.5</u> When $p = 1$, the test based on SS is the two-sided univariate Wilcoxon signed-rank test.

Proof of Lemma 5.1.5 Note that when $p = 1$,

$$\hat{p}_{ik,i'k'} = \begin{cases} 1 & \text{if } (X_i + X_k)(X_{i'} + X_{k'}) < 0 \\ \frac{1}{2} & \text{if } (X_i + X_k)(X_{i'} + X_{k'}) = 0 \\ 0 & \text{if } (X_i + X_k)(X_{i'} + X_{k'}) > 0 \end{cases},$$

and thus

$$\cos(\pi\hat{p}_{ik,i'k'}) = \begin{cases} -1 & \text{if } (X_i + X_k)(X_{i'} + X_{k'}) < 0 \\ 0 & \text{if } (X_i + X_k)(X_{i'} + X_{k'}) = 0 \\ 1 & \text{if } (X_i + X_k)(X_{i'} + X_{k'}) > 0 \end{cases}.$$

Hence, we can write

$$\cos(\pi\hat{p}_{ik,i'k'}) = \text{sign }(X_i + X_k)\cdot\text{sign }(X_{i'} + X_{k'})$$

where $\text{sign}(x) = -1$ if $x < 0$, 0 if $x = 0$, and 1 if $x > 0$.

Therefore we have

$$\begin{aligned} SS &= \frac{4}{n(n+1)^2} \sum_{i\leq k} \sum_{i'\leq k'} \cos(\pi\,\hat{p}_{ik,i'k'}) \\ &= \frac{4}{n(n+1)^2} \sum_{i\leq k} \sum_{i'\leq k'} \text{sign }(X_i + X_k)\cdot\text{sign }(X_{i'} + X_{k'}) \\ &= \frac{4}{n(n+1)^2} \{ \sum_{i\leq k} \text{sign }(X_i + X_k)\}^2. \qquad (5.1.6) \end{aligned}$$

Noting that the Wilcoxon signed rank test W^+ is equal to the number of positive Walsh averages $((X_i + X_k)/2,\ 1 \leq i \leq k \leq n)$ (See, e.g., Randles and Wolfe [1979], p.57 and p.83), thus we have

$$\sum_{i\leq k} \text{sign }(X_i + X_k) = W^+ - W^-,$$

where W^- is equal to the number of negative Walsh averages $((X_i + X_k)/2,\ 1 \le i \le k \le n)$.

Now using the fact that with probability one

$$W^+ - W^- = \frac{n(n+1)}{2},$$

expression (5.1.6) is equivalent to

$$\begin{aligned} SS &= \frac{4}{n(n+1)^2}\{W^+ - W^-\}^2 \\ &= \frac{4}{n(n+1)^2}\{2W^+ - \frac{n(n+1)}{2}\}^2 \\ &= \frac{16}{n(n+1)^2}\{W^+ - \frac{n(n+1)}{4}\}^2 \\ &= \frac{(W^+ - \frac{n(n+1)}{4})^2}{\frac{n(n+1)(2n+1)}{24}} \cdot \frac{\frac{n(n+1)(2n+1)}{24}}{\frac{n(n+1)^2}{16}} \\ &= \frac{(W^+ - E_{H_0}(W^+))^2}{Var_{H_0}(W^+)} \cdot \frac{2(2n+1)}{3(n+1)}, \end{aligned}$$

since

$$E_{H_0}(W^+) = \frac{n(n+1)}{4} \text{ and } Var_{H_0}(W^+) = \frac{n(n+1)(2n+1)}{24}.$$

(See, e.g., Randles and Wolfe [1979], p.56.) This completes the proof.

Note that, when $p = 1$,

$$\frac{3}{4} SS \xrightarrow{d} \chi_1^2 \text{ under } H_0 \text{ as } n \to \infty.$$

This follows from the fact that

$$\frac{W^+ - E_{H_0}(W^+)}{Var_{H_0}(W^+)} \xrightarrow{d} N(0,1) \text{ under } H_0$$

(See Randles and Wolfe [1979], p.85), and $\frac{2(2n+1)}{3(n+1)} \to \frac{3}{4}$ as $n \to \infty$.

5.2 Some Intermediate Results

In this section we present some basic and important results which will be useful in the next section for finding the limiting distribution of SS under H_0. Here we discuss some properties involving the sum observations $\underline{X}_i + \underline{X}_k$, $1 \le i \le k \le n$, when the original sample $\underline{X}_i$, $1 \le i \le n$, is from the family of elliptically symmetric distributions. The first one is about the sum and difference of two i.i.d. spherically symmetric random vectors, a special case of elliptically symmetric distributions.

Theorem 5.2.1 Let $\underline{X}_1, \underline{X}_2$ be i.i.d. spherically symmetric random vectors. Then $\underline{X}_1+\underline{X}_2$ and $\underline{X}_1-\underline{X}_2$ are spherically symmetric random vectors.

Proof of Theorem 5.2.1 Let $\underline{D}$ be any p×p orthogonal matrix. It suffices to prove that

$$\underline{D}(\underline{X}_1 \pm \underline{X}_2) \stackrel{d}{=} \underline{X}_1 \pm \underline{X}_2$$

(see, e.g., Muirhead [1982], p. 32), where $\stackrel{d}{=}$ is read “has the same distribution as”. Note that we have

$$\underline{X}_1 \stackrel{d}{=} \underline{D}\,\underline{X}_1 \text{ and } \underline{X}_2 \stackrel{d}{=} \underline{D}\,\underline{X}_2,$$

since $\underline{X}_1, \underline{X}_2$ are spherically symmetric.

Hence, by the independence of $\underline{X}_1$ and $\underline{X}_2$, we have

$$(\underline{X}_1, \underline{X}_2) \stackrel{d}{=} (\underline{D}\,\underline{X}_1, \underline{D}\,\underline{X}_2).$$

Therefore, see, e.g., Randles and Wolfe (1979) , p. 16,

$$g(\underline{X}_1, \underline{X}_2) \stackrel{d}{=} g(\underline{D}\,\underline{X}_1, \underline{D}\,\underline{X}_2),$$

where $g(\underline{x}, \underline{y}) = \underline{x} + \underline{y}$.

Hence

$$\underline{X}_1 + \underline{X}_2 \stackrel{d}{=} \underline{D}\,\underline{X}_1 + \underline{D}\,\underline{X}_2 = \underline{D}(\underline{X}_1 + \underline{X}_2).$$

The difference is handled similarly. This completes the proof.

Next we consider the general case of two i.i.d. elliptically symmetric random vectors.

Theorem 5.2.2 Let $\underline{X}_1, \underline{X}_2$ be i.i.d. elliptically symmetric random vectors. Then $\underline{X}_1+\underline{X}_2$ and $\underline{X}_1-\underline{X}_2$ are elliptically symmetric random vectors.

Proof of Theorem 5.2.1 Since $\underline{X}_1, \underline{X}_2$ are i.i.d. elliptically symmetric random vectors, we can write

$$\underline{X}_1 = \underline{A}\,\underline{X}_1^* \text{ and } \underline{X}_2 = \underline{A}\,\underline{X}_2^*,$$

where $\underline{X}_1^*, \underline{X}_2^*$ are i.i.d. spherically symmetric random vectors and $\underline{A}$ is a pxp nonsingular matrix. Thus

$$\underline{X}_1 \pm \underline{X}_2 = \underline{A}\,\underline{X}_1^* \pm \underline{A}\,\underline{X}_2^* = \underline{A}(\underline{X}_1^* \pm \underline{X}_2^*).$$

This completes the proof since $\underline{X}_1^* \pm \underline{X}_2^*$ are spherically symmetric random vectors, as shown in Theorem 5.2.1.

Define

$$\underline{h}(\underline{X}_1, \underline{X}_2) = \frac{\underline{X}_1+\underline{X}_2}{\| \underline{X}_1+\underline{X}_2 \|}, \tag{5.2.3}$$

where

$$\| \underline{t} \| = \sqrt{\underline{t}^T \underline{t}} = (\sum_{j=1}^{p} t_j^2)^{1/2}, \underline{t} = (t_1, \ldots, t_p).$$

We examine the property of $\underline{h}(\underline{X}_1, \underline{X}_2)$ in the next theorem.

Theorem 5.2.4 Let $\underline{X}_1, \underline{X}_2$ be i.i.d. spherically symmetric random vectors. Then $\underline{h}(\underline{X}_1, \underline{X}_2)$ is uniformly distributed on the p-dimentional unit sphere S_p.

Proof of Theorem 5.2.4 Since $\underline{X}_1, \underline{X}_2$ are i.i.d. spherically symmetric random vectors, $\underline{X}_1+\underline{X}_2$ is also spherically symmetric, as shown in Theorem 5.2.1. Thus it follows that

$$\underline{h}(\underline{X}_1, \underline{X}_2) = \frac{\underline{X}_1+\underline{X}_2}{\| \underline{X}_1+\underline{X}_2 \|} \sim U(S_p).$$

(See, e.g., Muirhead [1982], p. 38, Theorem 1.5.6.) The proof is complete.

Note that under the assumptions of Theorem 5.2.4

$$E_{H_0}[\underline{h}(\underline{X}_1, \underline{X}_2)] = \underline{0} = E_{\underline{X}_1}(E_{\underline{X}_2}[\underline{h}(\underline{X}_1, \underline{X}_2) \mid \underline{X}_1]).$$

Define

$$\underline{h}^*(\underline{X}_i) = E_{\underline{X}_k}[\underline{h}(\underline{X}_i, \underline{X}_k) \mid \underline{X}_i], \; 1 \le i \le k \le n. \tag{5.2.5}$$

We are interested in the behavior of $\underline{h}^*(\underline{X}_i)$, $1 \le i \le n$.

Theorem 5.2.6 Let $\underline{X}_1, \underline{X}_2$ be i.i.d. spherically symmetric random vectors. Then $\underline{h}^*(\underline{X}_1)$ is a spherically symmetric random vector.

Proof of Theorem 5.2.6 Let $\underline{D}$ be any p×p orthogonal matrix. It suffices to show that

$$\underline{D}\,\underline{h}^*(\underline{X}_1) \stackrel{d}{=} \underline{h}^*(\underline{X}_1).$$

(See, e.g., Muirhead [1982], p. 32.) Note that

$$\begin{aligned}
\underline{D}\,\underline{h}^*(\underline{X}_1) &= \underline{D}\,E_{\underline{X}_2}[\underline{h}(\underline{X}_1, \underline{X}_2) \mid \underline{X}_1] \\
&= \underline{D}\,E_{\underline{X}_2}\Big[\frac{\underline{X}_1+\underline{X}_2}{\| \underline{X}_1+\underline{X}_2 \|} \mid \underline{X}_1\Big] \text{ (by expression (5.2.3))} \\
&= E_{\underline{X}_2}\Big[\frac{\underline{D}\,\underline{X}_1+\underline{D}\,\underline{X}_2}{\| \underline{X}_1+\underline{X}_2 \|} \mid \underline{X}_1\Big] \\
&= E_{\underline{X}_2}\Big[\frac{\underline{D}\,\underline{X}_1+\underline{D}\,\underline{X}_2}{\| \underline{D}\,\underline{X}_1+\underline{D}\,\underline{X}_2 \|} \mid \underline{X}_1\Big] \text{ (} \| \underline{D} \| = 1 \text{ since } \underline{D} \text{ is orthogonal)} \\
&= E_{\underline{X}_2}[\underline{h}(\underline{D}\,\underline{X}_1, \underline{D}\,\underline{X}_2) \mid \underline{X}_1] \text{ (by expression (5.2.3))} \\
&= E_{\underline{X}_2}[\underline{h}(\underline{D}\,\underline{X}_1, \underline{X}_2) \mid \underline{X}_1] \text{ (} \underline{D}\,\underline{X}_2 \stackrel{d}{=} \underline{X}_2 \text{ by spherical symmetry)} \\
&= \underline{h}^*(\underline{D}\,\underline{X}_1) \text{ (by expression (5.2.7))} \\
&\stackrel{d}{=} \underline{h}^*(\underline{X}_1) \text{ (because } \underline{D}\,\underline{X}_1 \stackrel{d}{=} \underline{X}_1 \text{ by spherical symmetry).} \qquad (5.2.7)
\end{aligned}$$

This completes the proof.

Hence, for i.i.d. spherically symmetric random vectors $\underline{X}_1$ and $\underline{X}_2$, it follows that we can express $\underline{h}^*(\underline{X}_1)$ as

$$\underline{h}^*(\underline{X}_1) = R_1^*\,\underline{U}_1^*,$$

where $\underline{U}_1^* \sim \text{Uniform}(S_p)$ is independent of R_1^*, a positive random variable. Defining

$$\tau^2 = E_{H_0}[(\underline{h}^*(\underline{X}_1))^T \underline{h}^*(\underline{X}_1)], \qquad (5.2.8)$$

we get

$$\begin{aligned}
\text{Var-Cov}_{H_0}[\underline{h}^*(\underline{X}_1)] &= E_{H_0}[(R_1^* \underline{U}_1^*)(R_1^* \underline{U}_1^*)^T] \text{ (since } E_{H_0}(\underline{h}^*(\underline{X}_1)) = \underline{0}\text{)} \\
&= E_{H_0}[(R_1^*)^2] E_{H_0}[\underline{U}_1^*(\underline{U}_1^*)^T] \\
&= \frac{1}{p} E_{H_0}[(R_1^*)^2] \cdot \underline{I}_p \\
&= \frac{\tau^2}{p} \underline{I}_p,
\end{aligned}$$

since

$$\tau^2 = E_{H_0}[(R_1^* \underline{U}_1^*)^T (R_1^* \underline{U}_1^*)] = E_{H_0}[(R_1^*)^2].$$

In summary we have the following result : if $\underline{X}_1, \ldots, \underline{X}_n$ are i.i.d. spherically symmetric random vectors, then $\underline{h}^*(\underline{X}_1), \ldots, \underline{h}^*(\underline{X}_n)$, defined in (5.2.5), are i.i.d. spherically symmetric random vectors with

$$E_{H_0}[\underline{h}^*(\underline{X}_1)] = \underline{0} \text{ and } \text{Var-Cov}_{H_0}[\underline{h}^*(\underline{X}_1)] = \frac{\tau^2}{p} \underline{I}_p,$$

where τ^2 is defined in (5.2.8). Expressing

$$\underline{X}_i = R_i \underline{U}_i \text{ and } \underline{h}^*(\underline{X}_i) = R_i^* \underline{U}_i^*, \; 1 \le i \le n,$$

we now try to relate R_i^* to R_i and $\underline{U}_i^*$ to $\underline{U}_i$. This is stated in the next theorem.

<u>Theorem 5.2.9</u> Let $\underline{X}_1, \underline{X}_2$ be i.i.d. spherically symmetric random vectors with $\underline{X}_i = R_i \underline{U}_i$, $i = 1, 2$, where $\underline{U}_i \sim$ Uniform(S_p) is independent of R_i. Then we can write

$$\underline{h}^*(\underline{X}_1) = R^*(R_1)\underline{U}_1, \tag{5.2.10}$$

where

$$R^*(R_1) = E_{R_2, U_{21}}\left\{ \frac{R_1/R_2+U_{21}}{\sqrt{(R_1/R_2+U_{21})^2+1-U_{21}^2}} \right\} \tag{5.2.11}$$

is independent of $\underline{U}_1$, and $\underline{U}_2 = (U_{21}, \ldots, U_{2p})^T$.

<u>Proof of Theorem 5.2.9</u> Let $\underline{X} = R\ \underline{U}$ and $\underline{h}^*(\underline{X}) = R^*\ \underline{U}^*$. First we want to show that $\underline{U}^* = \underline{U}$. Define

$$\underline{U}_0 = (1, 0, \ldots, 0)^T \text{ and } \underline{X}_0 = R\ \underline{U}_0.$$

Then

$$\underline{h}^*(\underline{X}_0) = E_{\underline{X}_2}\left[\frac{\underline{X}_0+\underline{X}_2}{\|\underline{X}_0+\underline{X}_2\|} \mid \underline{X}_0\right] \text{ (by expression (5.2.5))}$$

$$= E_{R_2, \underline{U}_2}\left\{ \begin{bmatrix} R+R_2U_{21} \\ R_2U_{22} \\ \vdots \\ R_2U_{2p} \end{bmatrix} \Big/ \sqrt{(R+R_2U_{21})^2+R_2^2U_{22}^2+\ldots+R_2^2U_{2p}^2} \right\}$$

$$= E_{R_2, \underline{U}_2}\left\{ \begin{bmatrix} R/R_2+U_{21} \\ U_{22} \\ \vdots \\ U_{2p} \end{bmatrix} \Big/ \sqrt{(R/R_2+U_{21})^2+U_{22}^2+\ldots+U_{2p}^2} \right\}$$

$$= E_{R_2, \underline{U}_2}\left\{ \begin{bmatrix} R/R_2+U_{21} \\ U_{22} \\ \vdots \\ U_{2p} \end{bmatrix} \Big/ \sqrt{(R/R_2+U_{21})^2+1-U_{21}^2} \right\}$$

$$(\text{since } \sum_{j=1}^{p} U_{2j}^2 = 1)$$

$$= E_{R_2, U_{21}}\left\{ E_{U_{22}, \ldots, U_{2p}}\left[\begin{bmatrix} R/R_2+U_{21} \\ U_{22} \\ \cdot \\ \cdot \\ \cdot \\ U_{2p} \end{bmatrix} \Big/ \sqrt{(R/R_2+U_{21})^2+1-U_{21}^2} \right.\right.$$

$$\left.\left. \Big| R_2, U_{21} \right] \right\}.$$

Noting that, given R_2 and U_{21}, the expected value of $U_{2j}, j = 2, \ldots, p$, is zero and thus we write the above expression as

$$\underline{h}^*(\underline{X}_0) = E_{R_2, U_{21}}\left\{ \begin{bmatrix} R/R_2+U_{21} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \Big/ \sqrt{(R/R_2+U_{21})^2+1-U_{21}^2} \right\}$$

$$= E_{R_2, U_{21}}\left\{ \frac{R/R_2+U_{21}}{\sqrt{(R/R_2+U_{21})^2+1-U_{21}^2}} \right\} \cdot \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

$$= R^*(R)\, \underline{U}_0, \tag{5.2.12}$$

where R^* is as defined in (5.2.11). Let $\underline{D}$ be any pxp orthogonal matrix. Now we need to show that

$$\underline{h}^*(\underline{D}\, \underline{X}_0) = R^*(R) \cdot \underline{D}\, \underline{U}_0.$$

Note that

$$\underline{h}^*(\underline{D}\, \underline{X}_0) = E_{\underline{X}_2}[\underline{h}(\underline{D}\, \underline{X}_0, \underline{X}_2) \mid \underline{X}_0]$$

$$= E_{\underline{X}_2}\left[\frac{\underline{D}\, \underline{X}_0 + \underline{X}_2}{\| \underline{D}\, \underline{X}_0 + \underline{X}_2 \|} \mid \underline{X}_0 \right]$$

$$= E_{\underline{X}_2}\left[\frac{\underline{D}\,\underline{X}_0+\underline{D}\,\underline{X}_2}{\|\underline{D}\,\underline{X}_0+\underline{D}\,\underline{X}_2\|}\mid \underline{X}_0\right] \text{ (since } \underline{X}_2 \overset{d}{=} \underline{D}\,\underline{X}_2)$$

$$= E_{\underline{X}_2}\left[\frac{\underline{D}\,\underline{X}_0+\underline{D}\,\underline{X}_2}{\|\underline{X}_0+\underline{X}_2\|}\mid \underline{X}_0\right] \text{ } (\|\underline{D}\| = 1 \text{ since } \underline{D} \text{ is orthogonal})$$

$$= \underline{D}\,E_{\underline{X}_2}\left[\frac{\underline{X}_0+\underline{X}_2}{\|\underline{X}_0+\underline{X}_2\|}\mid \underline{X}_0\right]$$

$$= \underline{D}\,\underline{h}^*(\underline{X}_0) = \underline{D}\,R^*(R)\,\underline{U}_0 \text{ (follows from (5.2.12))}$$

$$= R^*(R)\cdot\underline{D}\,\underline{U}_0. \tag{5.2.13}$$

Using expressions (5.2.12) and (5.2.13), we have shown that whenever $\underline{X} = R\,\underline{U}$, $\underline{h}^*(\underline{X}) = R^*(R)\,\underline{U}$. The indenpendence of $R^*(R)$ and $\underline{U}$ follows from the independence of R and $\underline{U}$ and the fact that $R^*(R)$ depends on R only. This completes the proof.

All the above theorems and discussions lead to the following result. Let $\underline{X}_1, \ldots, \underline{X}_n$ be i.i.d. spherically symmetric random vectors with $\underline{X}_i = R_i\,\underline{U}_i$, $1 \le i \le n$, where $\underline{U}_i \sim U(S_p)$ is independent of R_i. Then $\underline{h}^*(\underline{X}_1), \ldots, \underline{h}^*(\underline{X}_n)$ are i.i.d. spherically symmetric random vectors with $\underline{h}^*(\underline{X}_i) = R^*(R_i)\,\underline{U}_i$, $1 \le i \le n$, $R^*(R_i)$ is independent of $\underline{U}_i$,

$$E_{H_0}[\underline{h}^*(\underline{X}_1)] = \underline{0} \text{ and } \text{Var-Cov}_{H_0}[\underline{h}^*(\underline{X}_1)] = \frac{\tau^2}{p}\underline{I}_p,$$

where $R^*(R_i)$ and τ^2 are defined in (5.2.11) and (5.2.8), respectively.

5.3 Asymptotic Null Distribution of SS

In this section we establish the null limiting distribution of SS under the class of elliptically symmetric distributions, with density function as defined in (3.2.1). Since the

test based on SS is invariant with respect to nonsingular linear transformations, it suffices to consider

$$\underline{X}_i = R_i \underline{U}_i, \; 1 \le i \le n, \tag{5.3.1}$$

where $\underline{U}_i$'s are i.i.d. Uniform(S_p) random vectors and are independent of R_i's, i.i.d. postive random vectors. As a first step, we now seek an asymptotic approximation for the test statistic based on SS, which possesses the following two properties under H_0 : (a) the difference between SS and the approximating statistic converges to zero in probability, and (b) the limiting distribution of the approximating statistic is easily established. One candidate satisfying these considerations is

$$SS^* = \frac{4p}{n(n+1)^2} \sum_{i \le k} \sum_{i' \le k'} \cos(\alpha_{ik,i'k'}), \tag{5.3.2}$$

where $\alpha ik,i'k'$ = the angle between $\underline{X}_i + \underline{X}_k$ and $\underline{X}_{i'} + \underline{X}_{k'}$. The first property of this approximating statistic is stated and proved in the next theorem.

Theorem 5.3.3 If the observations $\underline{X}_i$'s are as defined in (5.3.1) and H_0 is true, then

$$SS - SS^* \xrightarrow{P} 0 \text{ as } n \to \infty.$$

Proof of Theorem 5.3.3 In proving this result we will need to show

$$E_{H_0}[(SS - SS^*)^2] \to 0 \text{ as } n \to \infty.$$

Note that, using expressions (5.1.3) and (5.3.2),

$$E_{H_0}[(SS - SS^*)^2] = \frac{16p^2}{n^2(n+1)^4} \sum_{i \le k} \sum_{i' \le k'} \sum_{s \le t} \sum_{s' \le t'} E_{H_0}\Big\{ [\cos(\pi \hat{p}_{ik,i'k'}) - \cos(\alpha_{ik,i'k'})] \cdot [\cos(\pi \hat{p}_{st,s't'}) - \cos(\alpha_{st,s't'})] \Big\} \tag{5.3.4}$$

It can be easily shown that if (i, k) = (i', k') or (s, t) = (s', t'), then the term is zero. Also, if any one of the pairs (i, k), (i', k'), (s, t) or (s', t') is disjoint from the integers in the others, then the expected value is zero. The outline of this proof is given in Appendix B. An examination of the terms in (5.3.4) shows that, the number of terms with no disjoint pair is of the order n^6. This is explained as follows. Suppose pair (i, k) is not disjoint from the integers in the other pairs, then there exists a pair with at least one common integer as in (i, k). Without loss of generality, let us assume this pair is (i, k'). Now the pair (s, t) could be disjoint from pairs (i, k) or (i, k'), but must have at least one integer common with that of (s', t'). (Otherwise pair (s, t) will be disjoint.) Let us call pair (s', t') as (s, t'). Thus the number of terms with no disjoint pair is

$$\sum_{i\leq k}\sum_{i'\leq k'}\sum_{s\leq t}\sum_{s'\leq t'} 1 = \frac{n(n+1)}{2}\cdot n\cdot\frac{n(n+1)}{2}\cdot n \approx n^6.$$

Therefore we can bound (5.3.4) by

$$(\text{constant})\cdot E_{H_0}\left\{\left[\cos(\pi\,\hat{p}_{ik,i'k'}) - \cos(\alpha_{ik,i'k'})\right]\cdot\left[\cos(\pi\,\hat{p}_{st,s't'}) - \cos(\alpha_{st,s't'})\right]\right\}$$

$$\leq \quad (\text{constant})\cdot E_{H_0}\left\{\left|\pi\,\hat{p}_{ik,i'k'} - \alpha_{ik,i'k'}\right|\cdot\left|\pi\,\hat{p}_{st,s't'} - \alpha_{st,s't'}\right|\right\}.$$

Note that

$$\left|\pi\,\hat{p}_{ik,i'k'} - \alpha_{ik,i'k'}\right|\cdot\left|\pi\,\hat{p}_{st,s't'} - \alpha_{st,s't'}\right| \leq 2\cdot 2 = 4 \text{ a.e.}$$

Furthermore, the $\pi\,\hat{p}$'s are consisent estimators of their respective α's. Thus, by the Lebesgue Dominated Convergence Theorem, see, e.g., Chow and Teicher (1978), p. 99, we have

$$E_{H_0}\left\{\left|\pi\,\hat{p}_{ik,i'k'} - \alpha_{ik,i'k'}\right|\cdot\left|\pi\,\hat{p}_{st,s't'} - \alpha_{st,s't'}\right|\right\} \longrightarrow 0 \text{ as } n \to \infty.$$

Hence we have proved

$$E_{H_0}[(SS - SS^*)^2] \longrightarrow 0 \text{ as } n \to \infty.$$

This completes the proof, since convergence in quadratic mean implies convergence in probability.

The next theorem proves the limiting distribution of SS^* under H_0.

Theorem 5.3.5 If the observations $\underline{X}_i$, $1 \le i \le n$, are as defined in (5.3.1) and H_0 is true, then

$$\frac{SS^*}{4\tau^2} \xrightarrow{d} \chi^2_p \text{ as } n \to \infty,$$

where τ^2 is defined (5.2.8).

Proof of Theorem 5.3.5 Note that

$$\begin{aligned} \alpha_{ik,i'k'} &= \text{angle between } (\underline{X}_i+\underline{X}_k) \text{ and } (\underline{X}_{i'}+\underline{X}_{k'}) \\ &= \text{angle between } \frac{\underline{X}_i+\underline{X}_k}{\| \underline{X}_i+\underline{X}_k \|} \text{ and } \frac{\underline{X}_{i'}+\underline{X}_{k'}}{\| \underline{X}_{i'}+\underline{X}_{k'} \|}. \end{aligned}$$

Hence, we can write

$$\cos(\alpha_{ik,i'k'}) = \left(\frac{\underline{X}_i+\underline{X}_k}{\| \underline{X}_i+\underline{X}_k \|}\right)^T \cdot \left(\frac{\underline{X}_{i'}+\underline{X}_{k'}}{\| \underline{X}_{i'}+\underline{X}_{k'} \|}\right),$$

since $\frac{\underline{X}_i+\underline{X}_k}{\| \underline{X}_i+\underline{X}_k \|}$ has norm 1.

Now SS^* in (5.3.2) can be written as

$$\begin{aligned} SS^* &= \frac{4p}{n(n+1)^2} \sum_{i \le k} \sum_{i' \le k'} \left(\frac{\underline{X}_i+\underline{X}_k}{\| \underline{X}_i+\underline{X}_k \|}\right)^T \cdot \left(\frac{\underline{X}_{i'}+\underline{X}_{k'}}{\| \underline{X}_{i'}+\underline{X}_{k'} \|}\right) \\ &= \frac{4p}{n(n+1)^2} \left(\sum_{i \le k} \frac{\underline{X}_i+\underline{X}_k}{\| \underline{X}_i+\underline{X}_k \|}\right)^T \cdot \left(\sum_{i' \le k'} \frac{\underline{X}_{i'}+\underline{X}_{k'}}{\| \underline{X}_{i'}+\underline{X}_{k'} \|}\right) \end{aligned}$$

$$= \; p\cdot\Big(\frac{\sqrt{n}}{\frac{n(n+1)}{2}}\sum_{i\le k}\frac{\underline{X}_i+\underline{X}_k}{\|\underline{X}_i+\underline{X}_k\|}\Big)^T\cdot\Big(\frac{\sqrt{n}}{\frac{n(n+1)}{2}}\sum_{i'\le k'}\frac{\underline{X}_{i'}+\underline{X}_{k'}}{\|\underline{X}_{i'}+\underline{X}_{k'}\|}\Big)$$

$$= \; p\cdot\underline{R}_n^T\underline{R}_n, \tag{5.3.6}$$

where

$$\underline{R}_n = \frac{\sqrt{n}}{\frac{n(n+1)}{2}}\sum_{i\le k}\frac{\underline{X}_i+\underline{X}_k}{\|\underline{X}_i+\underline{X}_k\|}.$$

Moreover let us write

$$\underline{R}_n \; = \; \Big(\frac{\sqrt{n}}{\frac{n(n+1)}{2}}\sum_{i<k}\frac{\underline{X}_i+\underline{X}_k}{\|\underline{X}_i+\underline{X}_k\|} + \frac{\sqrt{n}}{\frac{n(n+1)}{2}}\sum_{i=k}\frac{\underline{X}_i+\underline{X}_k}{\|\underline{X}_i+\underline{X}_k\|}\Big)$$

$$= \; \sqrt{n}\,\Big(\frac{n-1}{n+1}\Big)\,\underline{U}_{1n} + \Big(\frac{2\sqrt{n}}{n+1}\Big)\,\underline{U}_{2n}, \tag{5.3.7}$$

where

$$\underline{U}_{1n} = \frac{1}{\binom{n}{2}}\sum_{i<k}\frac{\underline{X}_i+\underline{X}_k}{\|\underline{X}_i+\underline{X}_k\|} \text{ and } \underline{U}_{2n} = \frac{1}{n}\sum_{i=1}^{n}\frac{\underline{X}_i}{\|\underline{X}_i\|}.$$

Since, under H_0, $\frac{\underline{X}_i}{\|\underline{X}_i\|}$, $1 \le i \le n$, are i.i.d. Uniform (S_p) random vectors (as shown in Theorem 5.2.4), we have

$$\underline{U}_{2n} \xrightarrow{P} 0 \text{ as } n \to \infty.$$

Using the fact that $\frac{n-1}{n+1} \to 1$ and $\frac{2\sqrt{n}}{n+1} \to 0$ as $n \to \infty$, we see that, under H_0, expression (5.3.7) implies

$$\underline{R}_n - \sqrt{n}\,\underline{U}_{1n} \xrightarrow{P} 0 \text{ as } n \to \infty. \tag{5.3.8}$$

Combining expressions (5.3.6) and (5.3.8), now it suffices to find the limiting distribution of $\sqrt{n}\,\underline{U}_{1n}$ under H_0. Define

$$\underline{U}^*_{1n} = \frac{2}{n} \cdot \sum_{i=1}^{n} \underline{h}^*(\underline{X}_i), \tag{5.3.9}$$

where $\underline{h}^*$ is as defined in (5.2.5).

It follows from Theorem 3.3.13 (Randles and Wolfe [1979], p. 82) that, under H_0,

$$\sqrt{n}\,(\underline{U}_{1n} - \underline{U}^*_{1n}) \xrightarrow{P} 0 \text{ as } n \to \infty,$$

and hence expression (5.3.8) implies

$$\underline{R}_n - \sqrt{n}\;\underline{U}^*_{1n} \xrightarrow{P} 0 \text{ as } n \to \infty. \tag{5.3.10}$$

Note that, under H_0,

$$\sqrt{n}\;\underline{U}^*_{1n} = \sqrt{n}\,\Big(\frac{1}{n} \cdot \sum_{i=1}^{n} 2\underline{h}^*(\underline{X}_i)\Big),$$

where $\underline{h}^*(\underline{X}_i)$'s are i.i.d. with

$$E_{H_0}[\underline{h}^*(\underline{X}_1)] = \underline{0} \text{ and } Var_{H_0}[\underline{h}^*(\underline{X}_1)] = \frac{\tau^2}{p} \cdot \underline{I}_p$$

(see section 5.2). Hence by the multivariate central limit theorem we have, under H_0,

$$\sqrt{n}\;\underline{U}^*_{1n} \xrightarrow{d} N_p\Big(\underline{0}, \frac{4\tau^2}{p}\underline{I}_p\Big) \text{ as } n \to \infty. \tag{5.3.11}$$

Using expressions (5.3.10) and (5.3.11), it follows from Slutsky's Theorem that,

$$\underline{R}_n \xrightarrow{d} N_p\Big(\underline{0}, \frac{4\tau^2}{p}\underline{I}_p\Big) \text{ as } n \to \infty,$$

and hence

$$\frac{\sqrt{p}}{2\tau}\underline{R}_n \xrightarrow{d} N_p(\underline{0}, \underline{I}_p) \text{ as } n \to \infty.$$

Therefore, under H_0, using (5.3.6), we have

$$\frac{SS^*}{4\tau^2} = \left(\frac{\sqrt{p}}{2\tau}\underline{R}_n\right)^T\left(\frac{\sqrt{p}}{2\tau}\underline{R}_n\right) \xrightarrow{d} \chi_p^2 \text{ as } n \to \infty.$$

This completes the proof.

Now we are ready to state the theorem for the limiting distribution of SS under H_0.

Theorem 5.3.12 If the observations $\underline{X}_i$, $1 \le i \le n$, are as defined in (5.3.1) and H_0 is true, then

$$\frac{SS}{4\tau^2} \xrightarrow{d} \chi_p^2 \text{ as } n \to \infty,$$

where τ^2 is defined (5.2.8).

Proof of Theorem 5.3.12 With Theorems 5.3.3 and 5.3.5 established, the result follows directly from the Slutsky's Theorem.

Note that when $p = 1$, $\tau^2 = 1/3$. Thus when $p = 1$,

$$\frac{SS}{4\tau^2} = \frac{3}{4}\, SS \xrightarrow{d} \chi_1^2 \text{ as } n \to \infty.$$

Finally, to perform the test will require a consistent estimator of τ^2, say $\hat{\tau}^2$. The test will then be based on the fact that under H_0,

$$\frac{SS}{4\hat{\tau}^2} \xrightarrow{d} \chi_p^2 \text{ as } n \to \infty.$$

For consistent estimator of τ^2, we consider

$$\hat{\tau}^2 = \frac{2}{n(n-1)(n-2)} \sum_{i=1}^{n} \sum_{\substack{k<k' \\ k\, k' \neq i}} \cos(\pi\, \hat{p}_{ik,ik'}), \tag{5.3.13}$$

where $\hat{p}_{ik,ik'}$ is as defined in (5.1.4). The consistency is proved in the next theorem.

Theorem 5.3.14 Let τ^2 be as defined in (5.2.8). Then $\hat{\tau}^2$ is a consistent estimator of τ^2 under H_0.

Proof of Theorem 5.3.14 Define

$$\tilde{\tau}^2 = \frac{2}{n(n-1)(n-2)} \sum_{i=1}^{n} \sum_{\substack{k<k' \\ k\, k' \neq i}} \cos(\alpha_{ik,ik'}).$$

Using the same arguments as in the proof of theorem 5.3.3, it can seen that under H_0,

$$\hat{\tau}^2 - \tilde{\tau}^2 \xrightarrow{P} 0 \text{ as } n \to \infty.$$

So we can conclude our proof by showing, under H_0,

$$\tilde{\tau}^2 - \tau^2 \xrightarrow{P} 0 \text{ as } n \to \infty.$$

Note that

$$\tau^2 = E_{H_0}[(\underline{h}^*(\underline{X}_1))^T \underline{h}^*(\underline{X}_1)] \text{ (by expression (5.2.8))}$$

$$= E_{H_0}\left[\frac{(\underline{X}_1+\underline{X}_2)^T(\underline{X}_1+\underline{X}_3)}{\| \underline{X}_1+\underline{X}_2 \| \| \underline{X}_1+\underline{X}_3 \|} \right] \text{ (by expressions (5.2.3) and (5.2.8))}$$

$$= E_{H_0}\left[\frac{(\underline{X}_i+\underline{X}_k)^T(\underline{X}_i+\underline{X}_{k'})}{\|\underline{X}_i+\underline{X}_k\|\,\|\underline{X}_i+\underline{X}_{k'}\|}\right],\ 1 \le i \le n,\ k < k',\ k \neq i,\ k' \neq i.$$

$$= E_{H_0}[g(\underline{X}_i, \underline{X}_k, \underline{X}_{k'})], \qquad (5.3.15)$$

where

$$g(\underline{X}_i, \underline{X}_k, \underline{X}_{k'}) = \frac{(\underline{X}_i+\underline{X}_k)^T(\underline{X}_i+\underline{X}_{k'})}{\|\underline{X}_i+\underline{X}_k\|\,\|\underline{X}_i+\underline{X}_{k'}\|}$$

satisfies $g(\underline{X}_i, \underline{X}_k, \underline{X}_{k'}) = g(\underline{X}_i, \underline{X}_{k'}, \underline{X}_k)$. Also we can write

$$\tilde{\tau}^2 = \frac{2}{n(n-1)(n-2)} \sum_{i=1}^{n} \sum_{\substack{k<k' \\ k\ k' \neq i}} \cos(\alpha_{ik,ik'})$$

$$= \frac{2}{n(n-1)(n-2)} \sum_{i=1}^{n} \sum_{\substack{k<k' \\ k\ k' \neq i}} \frac{(\underline{X}_i+\underline{X}_k)^T(\underline{X}_i+\underline{X}_{k'})}{\|\underline{X}_i+\underline{X}_k\|\,\|\underline{X}_i+\underline{X}_{k'}\|}$$

$$= \frac{2}{n(n-1)(n-2)} \sum\sum\sum_{i<k<k'} \left\{ \frac{(\underline{X}_i+\underline{X}_k)^T(\underline{X}_i+\underline{X}_{k'})}{\|\underline{X}_i+\underline{X}_k\|\,\|\underline{X}_i+\underline{X}_{k'}\|} \right.$$

$$\left. + \frac{(\underline{X}_k+\underline{X}_i)^T(\underline{X}_k+\underline{X}_{k'})}{\|\underline{X}_k+\underline{X}_i\|\,\|\underline{X}_k+\underline{X}_{k'}\|} + \frac{(\underline{X}_{k'}+\underline{X}_i)^T(\underline{X}_{k'}+\underline{X}_k)}{\|\underline{X}_{k'}+\underline{X}_i\|\,\|\underline{X}_{k'}+\underline{X}_k\|} \right\}$$

$$= \frac{1}{\binom{n}{3}} \sum\sum\sum_{i<k<k'} \frac{1}{3} \left\{ \frac{(\underline{X}_i+\underline{X}_k)^T(\underline{X}_i+\underline{X}_{k'})}{\|\underline{X}_i+\underline{X}_k\|\,\|\underline{X}_i+\underline{X}_{k'}\|} \right.$$

$$\left. + \frac{(\underline{X}_k+\underline{X}_i)^T(\underline{X}_k+\underline{X}_{k'})}{\|\underline{X}_k+\underline{X}_i\|\,\|\underline{X}_k+\underline{X}_{k'}\|} + \frac{(\underline{X}_{k'}+\underline{X}_i)^T(\underline{X}_{k'}+\underline{X}_k)}{\|\underline{X}_{k'}+\underline{X}_i\|\,\|\underline{X}_{k'}+\underline{X}_k\|} \right\}$$

$$= \frac{1}{\binom{n}{3}} \sum\sum\sum_{i<k<k'} g^*(\underline{X}_i, \underline{X}_k, \underline{X}_{k'}),$$

where

$$g^*(\underline{X}_i, \underline{X}_k, \underline{X}_{k'}) = \frac{1}{3} \left\{ \frac{(\underline{X}_i+\underline{X}_k)^T(\underline{X}_i+\underline{X}_{k'})}{\|\underline{X}_i+\underline{X}_k\|\,\|\underline{X}_i+\underline{X}_{k'}\|} + \frac{(\underline{X}_k+\underline{X}_i)^T(\underline{X}_k+\underline{X}_{k'})}{\|\underline{X}_k+\underline{X}_i\|\,\|\underline{X}_k+\underline{X}_{k'}\|} \right.$$

$$+\frac{(\underline{X}_{k'}+\underline{X}_i)^T(\underline{X}_{k'}+\underline{X}_k)}{\|\underline{X}_{k'}+\underline{X}_i\| \, \|\underline{X}_{k'}+\underline{X}_k\|}\Bigg\} \tag{5.3.16}$$

$$= \frac{1}{3}\left\{ \underline{g}(\underline{X}_i, \underline{X}_k, \underline{X}_{k'}) + \underline{g}(\underline{X}_k, \underline{X}_i, \underline{X}_{k'}) + \underline{g}(\underline{X}_{k'}, \underline{X}_i, \underline{X}_k)\right\}$$

is symmetric in its arguments. Thus, by Corollary 3.2.5, Randles and Wolfe (1979), p. 71, we have, under H_0,

$$\tilde{\tau}^2 - \tau^2 \xrightarrow{P} 0 \text{ as } n \to \infty.$$

5.4 Asymptotic Distribution of SS under Contiguous Alternatives

In this section we establish the asymptotic distribution of SS under a sequence of alternatives approaching the null hypothesis $H_0 : \underline{\theta}^* = \underline{0}$ for a specific class of elliptically symmetric distributions. As a first step, let us assume $\underline{X}_1, \ldots, \underline{X}_n$ are i.i.d. as $\underline{X} = (X_1, \ldots, X_p)^T$, where $\underline{X}$ is elliptically symmetric with a density function $f_{\underline{X}}$ of the form

$$f_{\underline{X}}(\underline{x}) = K_p|\underline{\Sigma}|^{-1/2}\exp\{-[(\underline{x} - \underline{\theta}^*)^T\underline{\Sigma}^{-1}(\underline{x} - \underline{\theta}^*)/C_0]^v\},\ \underline{x} \in R^p, \tag{5.4.1}$$

where K_p, C_0 are as defined in (2.3.2), $\underline{\theta}^*$ is the point of symmetry, and $\underline{\Sigma}$ is the variance-covariance matrix. Since both SS and Hotelling's T^2 are affine-invariant, we can, without loss of generality, assume that $\underline{\Sigma} = \underline{I}_p$, the p×p identity matrix. Thus, under H_0, $\underline{X}_1, \ldots, \underline{X}_n$ are i.i.d. with density of the form

$$f_{\underline{X}}(\underline{x}) = K_p \cdot \exp\{-[(\underline{x}^T\underline{x})/C_0]^v\}. \tag{5.4.2}$$

Under the sequence of alternatives considered in section 2.3, $\underline{X}_1, \ldots, \underline{X}_n$ are i.i.d. with density $f_{\underline{X}}(\underline{x}-\underline{c}n^{-1/2})$, where $f_{\underline{X}}$ is given in (5.4.2) and $\underline{c} \in R^p-\{0\}$ is arbitrary, but fixed. Substituting p+1 for p in the proof of Appendix A, we have shown that if $4\nu + p > 2$, then

$$0 < I_{\underline{c}}(f_{\underline{X}}) = \int \left[\frac{\underline{c}^T \partial f_{\underline{X}}(\underline{x})}{f_{\underline{X}}(\underline{x})} \right]^2 f_{\underline{X}}(\underline{x}) d\underline{x} < \infty \quad \text{for all } \underline{c} \neq \underline{0}. \tag{5.4.3}$$

Thus, when $4\nu + p > 2$, the previous results in conjunction with LeCam's Theorems on contiguity (see Hájek and Sidák [1967] pages 212-213) can be used to establish the asymptotic distribution of SS under the sequence of contiguous alternatives. This is stated in the next theorem.

Theorem 5.4.4 Let $\underline{X}_1, \ldots, \underline{X}_n$ be i.i.d. from an elliptically symmetric distribution with density function $f_{\underline{X}}$ given in (5.4.1) with $\underline{\Sigma} = \underline{I}_p$, the p×p identity matrix. If $4\nu + p > 2$, then under the sequence of contiguous alternatives for which $\underline{X}$ has density of the form $f_{\underline{X}}(\underline{x}-\underline{c}n^{-1/2})$, defined in (5.4.2), we have

$$\frac{SS}{4\tau^2} \xrightarrow{d} \chi_p^2\left(\frac{4\nu^2}{\tau^2 p C_0^{2\nu}} \left\{ E_{H_0}\left[R^{2\nu-1} R^*(R) \right] \right\}^2 \underline{c}^T \underline{c} \right), \text{ as } n \to \infty, \tag{5.4.5}$$

where τ^2 and $R^*(R)$ are as defined in (5.2.8) and (5.2.11), respectively.

Proof of Theorem 5.4.4 Define

$$T_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\frac{\underline{c}^T \partial f_{\underline{X}}(\underline{X}_i)}{f_{\underline{X}}(\underline{X}_i)} \right],$$

$$\underline{U}_{1n}^* = \frac{2}{n} \cdot \sum_{i=1}^{n} \underline{h}^*(\underline{X}_i),$$

and

$$S_n = \sqrt{n}\,\underline{\lambda}^T \underline{U}^*_{1n},$$

where $\underline{c}, \underline{\lambda} \in R^p - \{\underline{0}\}$, and $\underline{h}^*$ is defined in (5.2.5). Here $\underline{X}_i = R_i\,\underline{U}_i$, where $\underline{U}_i$ is distributed uniformly on the p-dimensional unit-sphere independent of R_i, and $R_i^2 = \underline{X}_i^T\underline{X}_i$,

thus we can write

$$T_n = \frac{2\nu}{C_0^\nu\sqrt{n}} \sum_{i=1}^{n} \underline{c}^T\underline{X}_i(\underline{X}_i^T\underline{X}_i)^{\nu-1}$$

$$= \frac{2\nu}{C_0^\nu\sqrt{n}} \sum_{i=1}^{n} \underline{c}^T R_i^{2\nu-1}\underline{U}_i.$$

Thus, under H_0,

$$\begin{bmatrix} S_n \\ T_n \end{bmatrix} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \begin{bmatrix} 2\underline{\lambda}^T\underline{h}^*(\underline{X}_i) \\ \frac{2\nu}{C_0^\nu}\underline{c}^T R_i^{2\nu-1}\,\underline{U}_i \end{bmatrix}$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \begin{bmatrix} 2\underline{\lambda}^T R^*(R_i)\,\underline{U}_i \\ \frac{2\nu}{C_0^\nu}\underline{c}^T R_i^{2\nu-1}\,\underline{U}_i \end{bmatrix}$$

(since $\underline{h}^*(\underline{X}_i) = R^*(R_i)\,\underline{U}_i$, as proved in Theorem 5.2.9)

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \begin{bmatrix} A_i \\ B_i \end{bmatrix} \text{(say)},$$

where

$$A_i = 2\underline{\lambda}^T R^*(R_i)\,\underline{U}_i \text{ and } B_i = \frac{2\nu}{C_0^\nu}\underline{c}^T R_i^{2\nu-1}\,\underline{U}_i.$$

Note that, under H_0, $\begin{bmatrix} A_i \\ B_i \end{bmatrix}$'s are i.i.d. with

$$E\begin{bmatrix}A_i\\B_i\end{bmatrix}=\underline{0} \quad \text{and} \quad Var\begin{bmatrix}A_i\\B_i\end{bmatrix}=\begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{bmatrix},$$

where

$$\begin{aligned}
\sigma_{11} &= E(A_i^2)\\
&= E(2\underline{\lambda}^T R^*(R_i)\, \underline{U}_i \cdot 2R^*(R_i)\, \underline{U}_i^T \underline{\lambda})\\
&= 4E\{[R^*(R_i)]\}^2 \cdot \underline{\lambda}^T E(\underline{U}_i \underline{U}_i^T)\underline{\lambda}\\
&= \frac{4}{p} E\{[R^*(R_i)]\}^2 \cdot \underline{\lambda}^T \underline{\lambda},
\end{aligned}$$

$$\begin{aligned}
\sigma_{22} &= E(B_i^2)\\
&= \frac{4v^2}{C_0^{2v}} E(R_i^{4v-2}) \cdot \underline{c}^T E(\underline{U}_i \underline{U}_i^T)\underline{c}\\
&= \frac{4v^2}{pC_0^{2v}} E(R_i^{4v-2}) \cdot \underline{c}^T \underline{c},
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{12} &= E(A_i B_i)\\
&= E(2\underline{\lambda}^T R^*(R_i)\, \underline{U}_i \cdot \frac{2v}{C_0^v} R_i^{2v-1}\, \underline{U}_i^T \underline{c})\\
&= \frac{4v}{C_0^v} E(R_i^{2v-1} R^*(R_i)) \cdot \underline{\lambda}^T E(\underline{U}_i \underline{U}_i^T)\underline{c}\\
&= \frac{4v}{pC_0^v} E[R_i^{2v-1} R^*(R_i)] \cdot \underline{\lambda}^T \underline{c}.
\end{aligned}$$

Thus, under H_0, see, e.g., Serfling(1980), Theorem B, p. 28,

$$\begin{bmatrix} S_n \\ T_n \end{bmatrix} \xrightarrow{d} N_2\left(\underline{0}, \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}\right) \text{ as } n \to \infty,$$

where σ_{11}, σ_{22}, and σ_{12} are as defined above. Applying LeCam's Theorems on contiguity (see Hájek and Sidák [1967], p. 212-213), if $4\nu + p > 2$, then we have, under the sequence of alternatives,

$$S_n \xrightarrow{d} N(\sigma_{12}, \sigma_{11}) \text{ as } n \to \infty.$$

That is, under the sequence of alternatives,

$$S_n = \sqrt{n}\,\underline{\lambda}^T \underline{U}^*_{1n} \xrightarrow{d} N\left(\frac{4\nu}{pC_0^\nu} E[R_i^{2\nu-1}R^*(R_i)]\cdot\underline{\lambda}^T\underline{c},\ \frac{4\tau^2}{p}\cdot\underline{\lambda}^T\underline{\lambda}\right) \text{ as } n \to \infty$$

since

$$E\{[R^*(R_i)]\}^2 = \tau^2.$$

Therefore, under the sequence of alternatives,

$$\sqrt{n}\,\underline{U}^*_{1n} \xrightarrow{d} N_p\left(\frac{4\nu}{pC_0^\nu} E[R_i^{2\nu-1}R^*(R_i)]\cdot\underline{c},\ \frac{4\tau^2}{p}\cdot\underline{I}_p\right) \text{ as } n \to \infty,$$

and hence

$$\frac{\sqrt{p}}{2\tau}\sqrt{n}\,\underline{U}^*_{1n} \xrightarrow{d} N_p\left(\frac{2\nu}{\sqrt{p}\tau C_0^\nu} E[R_i^{2\nu-1}R^*(R_i)]\cdot\underline{c},\ \underline{I}_p\right) \text{ as } n \to \infty.$$

Recall that

$$\underline{R}_n - \sqrt{n}\,\underline{U}^*_{1n} \xrightarrow{P} 0 \text{ as } n \to \infty \quad \text{and } SS^* = p\cdot\underline{R}_n^T\underline{R}_n$$

(see expressions (5.3.8) and (5.3.6), respectively). Using the Slutsky's Theorem we see that $\underline{R}_n$ and $\sqrt{n}\,\underline{U}_{1n}^*$ have the same limiting distribution, and thus, under the sequence of alternatives, we have,

$$\frac{SS^*}{4\tau^2} = \frac{p}{4\tau^2}\underline{R}_n^T\underline{R}_n$$

$$= \left(\frac{\sqrt{p}}{2\tau}\underline{R}_n\right)^T\left(\frac{\sqrt{p}}{2\tau}\underline{R}_n\right)$$

$$\xrightarrow{d} \chi_p^2\left(\frac{4\nu^2}{\tau^2 p C_0^{2\nu}}\left\{E_{H_0}\left[R^{2\nu-1}R^*(R)\right]\right\}^2\underline{c}^T\underline{c}\right), \text{ as } n \to \infty.$$

Therefore, under the sequence of contiguous alternatives,

$$\frac{SS}{4\tau^2} \xrightarrow{d} \chi_p^2\left(\frac{4\nu^2}{\tau^2 p C_0^{2\nu}}\left\{E_{H_0}\left[R^{2\nu-1}R^*(R)\right]\right\}^2\underline{c}^T\underline{c}\right), \text{ as } n \to \infty,$$

since

$$\frac{SS}{4\tau^2} - \frac{SS^*}{4\tau^2} \xrightarrow{P} 0 \text{ as } n \to \infty$$

(proved in Theorem 5.3.3). This completes the proof.

Finally, we establish the Pitman asymptotic relative efficiency of the test based on SS relative to Hotelling's T^2 in the next theorem.

Theorem 5.4.6 Let $\underline{X}_1, \ldots, \underline{X}_n$ be i.i.d. from an elliptically symmetric distribution with density function $f_{\underline{X}}$ given in (5.4.1) with $\underline{\Sigma} = \underline{I}_p$, the p×p identity matrix. If $4\nu + p > 2$,

then the Pitman asymptotic relative efficiency of the test based on SS relative to Hotelling's T^2 is

$$\text{ARE}\left(\frac{SS}{4\tau^2}, T^2\right) = \frac{4\nu^2}{\tau^2 p C_0^{2\nu}} \left\{ E_{H_0}\left[R^{2\nu-1}R^*(R)\right] \right\}^2, \tag{5.4.7}$$

where τ^2 and $R^*(R)$ are as defined in (5.2.8) and (5.2.11), respectively.

Proof of Theorem 5.4.6 Since under the sequence of contiguous alternatives,

$$T^2 \xrightarrow{d} \chi_p^2(\underset{\sim}{c}^T \underset{\sim}{c}), \text{ as } n \to \infty,$$

the result follows by taking the ratio of the noncentrality parameter of the test SS to the noncentrality parameter of the test T^2.

5.5 Numerical Evaluation of ARE($SS/4\tau^2$, T^2)

In this section I describe the numerical evaluation of the ARE expression given in Theorem 5.4.6. All calculations were performed with a Research Computing Initiative (RCI) account offered by the Northeast Regional Data Center (NERDC), running on an IBM computer using fortran 77. A subroutine called ADAPT (see Genz and Malik [1980]) was used for numerical integration over an N-dimensional rectangular region.

Now we briefly describe how the calculations were performed. First note that, following some standard transformations and arguments, we can express

$$E_{H_0}\left[R^{2\nu-1}R^*(R)\right] = \frac{K_p^2 \pi^p C_0^{p+\nu-1/2}}{\nu^2 \Gamma^2(\frac{p}{2})} \int_0^1 \int_0^1 f(x, y)\, dx dy, \tag{5.5.1}$$

and

$$\tau^2 = \left(\frac{K_p \pi^{p/2} C_0^{p/2}}{v \cdot \Gamma(\frac{p}{2})}\right)^3 \int_0^1 \int_0^1 \int_0^1 g(x, y, z)\, dxdydz, \tag{5.5.2}$$

where

$$f(x, y) = k\left(\left[\frac{(-\ell n x)}{(-\ell n y)}\right]^{\frac{1}{2v}}\right) \cdot (-\ell n x)^{\frac{p-1}{2v}} (-\ell n y)^{\frac{p-2v}{2v}},$$

$$g(x, y, z) = k\left(\left[\frac{(-\ell n x)}{(-\ell n y)}\right]^{\frac{1}{2v}}\right) \cdot k\left(\left[\frac{(-\ell n x)}{(-\ell n z)}\right]^{\frac{1}{2v}}\right) \cdot (-\ell n x)^{\frac{p-2v}{2v}} (-\ell n y)^{\frac{p-2v}{2v}} (-\ell n z)^{\frac{p-2v}{2v}},$$

$$k(s) = \int_0^\pi \frac{s+\cos(\theta)}{(s^2+2s\cdot\cos(\theta)+1)^{1/2}} \cdot \frac{\sin^{p-2}(\theta)\Gamma(\frac{p}{2})}{\Gamma(\frac{p-1}{2})\pi^{1/2}}\, d\theta, \tag{5.5.3}$$

K_p and C_0 are as defined in (2.3.2), and ℓn is the natural logarithm. Using expressions (5.5.1) and (5.5.2), we can simplify expression (5.4.7) to

$$ARE(SS/4\tau^2, T^2) = \text{constant} \cdot \frac{\left[\int_0^1 \int_0^1 f(x, y) dxdy\right]^2}{\int_0^1 \int_0^1 \int_0^1 g(x, y, z) dxdydz}, \tag{5.5.4}$$

where

$$\text{constant} = \frac{4v^2 \Gamma(\frac{p+2}{2v})}{p^2 \Gamma^2(\frac{p}{2v})}, \tag{5.5.5}$$

and f, g are as defined in (5.5.3). Now using subroutine ADAPT with a relative error ESTREL, we can approximate $\int_0^1 \int_0^1 f(x, y)dxdy$ by VAL satisfying

$$\left| \int_0^1 \int_0^1 f(x, y)\, dxdy - VAL \right| \le ERROR,$$

where ERROR = VAL × ESTREL. Hence we approximate $[\int_0^1\int_0^1 f(x, y)\,dxdy]^2$ by $(VAL)^2$

with the maximum error

$$\text{ERROR1} = 2\cdot\text{VAL}\cdot\text{ERROR} + (\text{ERROR})^2. \tag{5.5.6}$$

Also, we approximate $\int_0^1\int_0^1\int_0^1 g(x, y, z)\,dxdydz$ by TAU satisfying

$$\left|\int_0^1\int_0^1\int_0^1 g(x, y, z)\,dxdydz - \text{TAU}\right| \le \text{ERROR2}, \tag{5.5.7}$$

where ERROR2 = TAU×ESTREL. So we can write expression (5.5.4) as

$$\text{ARE}(SS/4\tau^2, T^2) = \text{constant}\cdot\left(\frac{(\text{VAL})^2 \pm \text{ERROR1}}{\text{TAU} \ \pm \text{ERROR2}}\right),$$

and hence approximate $\text{ARE}(SS/4\tau^2, T^2)$ by constant·$((\text{VAL})^2/\text{TAU})$ with the maximum error

$$\text{ERREST} = \text{constant}\cdot\frac{\text{ERROR1}\cdot\text{TAU} + \text{ERROR2}\cdot(\text{VAL})^2}{\text{TAU}(\text{TAU} - \text{ERROR2})}\ . \tag{5.5.8}$$

Note that when p = 1, using the expression (5.4.6) on p.166 of Randles and Wolfe (1979), the ARE expression in (5.4.7) can be simplified to

$$\text{ARE}(SS/4\tau^2, T^2) = \frac{12\nu^2\Gamma(\frac{3}{2\nu})}{2^{1/\nu}\Gamma^3(\frac{1}{2\nu})}\ . \tag{5.5.9}$$

In Table 5.1 we present the asymptotic relative efficiency of the interdirection signed sum test relative to Hotelling's T^2 for selected values of ν and p. When p = 1, the ARE's are evaluated from expression (5.5.9). For p = 2 to 5, and thus 4ν+p > 2 for all ν > 0, the ARE's are numerically evaluated using expressions (5.5.4) to (5.5.7). When the underlying population is multivariate normal (ν = 1.0), Hotelling's T^2 performs better, yet

the test based on SS appears to be quite competitive. For lighted-tailed distributions (ν = 2.0 to 5.0), Hotelling's T^2 still has better power. For heavy-tailed distributions (ν = .75, .50, .25, and .20), the interdirection signed sum test is more effective than Hotelling's T^2. In Table 5.2 we display the values of ERREST, defined in (5.5.8), which bound the error in the estimates of ARE($SS/4\tau^2$, T^2) for p = 2 to 5.

Table 5.1

ARE ($SS/4\tau^2$, T^2)

p	ν								
	5.0	4.0	3.0	2.0	1.0	.75	.50	.25	.20
1	0.9074	0.8950	0.8810	0.8727	0.9549	1.0788	1.5000	5.6250	11.9486
2	0.8884	0.8729	0.8713	0.8829	0.9662	1.0536	1.2963	2.7615	4.1880
3	0.8737	0.8761	0.8821	0.8983	0.9735	1.0403	1.2107	2.0584	2.7412
4	0.8781	0.8903	0.8953	0.9119	0.9780	1.0327	1.1644	1.7340	2.1681
5	0.8964	0.9000	0.9069	0.9223	0.9814	1.0277	1.1360	1.5912	1.9023

Table 5.2

Error Estimate ERREST of ARE ($SS/4\tau^2$, T^2)

	ν								
p	5.0	4.0	3.0	2.0	1.0	.75	.50	.25	.20
2	0.0094	0.0061	0.0029	0.0011	0.0012	0.0013	0.0016	0.0033	0.0061
3	0.0226	0.0089	0.0025	0.0011	0.0011	0.0013	0.0018	0.0228	0.0570
4	0.0287	0.0085	0.0019	0.0003	0.0010	0.0012	0.0027	0.0376	0.1356
5	0.0024	0.0012	0.0011	0.0010	0.0013	0.0026	0.0067	0.0659	0.1752

APPENDIX A

In the context of theorem 2.3.10, we show that if $4\nu + p > 3$, then

$$0 < I_{\underline{c}}(f_{\underline{Z}}) = \int \left[\frac{\underline{c}^T \partial f_{\underline{Z}}(\underline{z})}{f_{\underline{Z}}(\underline{z})} \right]^2 f_{\underline{Z}}(\underline{z}) d\underline{z} < \infty \quad \text{for all } \underline{c} \neq \underline{0}.$$

Proof Under $H_0 : \underline{\theta} = \underline{0}$, taking $\underline{A}\,\underline{\Sigma}\,\underline{A}^T = I_{p-1}$, $\underline{Z}$ has density function

$$f_{\underline{Z}}(\underline{z}) = K_p \int_{-\infty}^{\infty} \exp\left\{ -\left[\frac{\underline{z}^T\underline{z}+s^2}{C_0}\right]^{\nu} \right\} ds,$$

and thus, following standard arguments, we have

$$\partial f_{\underline{Z}}(\underline{z}) = K_p \int_{-\infty}^{\infty} (-\nu)\left[\frac{\underline{z}^T\underline{z}+s^2}{C_0}\right]^{\nu-1} \left(\frac{2\underline{z}}{C_0}\right) \cdot \exp\left\{ -\left[\frac{\underline{z}^T\underline{z}+s^2}{C_0}\right]^{\nu} \right\} ds.$$

Therefore,

$$\begin{aligned} I_{\underline{c}}(f_{\underline{Z}}) &= E\left[\frac{\underline{c}^T \partial f_{\underline{Z}}(\underline{Z})}{f_{\underline{Z}}(\underline{Z})} \right]^2 \\ &= \left(\frac{-4\nu K_p}{C_0^{\nu}}\right)^2 E\left\{ \left(\frac{\underline{c}^T\underline{Z}}{f_{\underline{Z}}(\underline{Z})}\right) \int_0^{\infty} \left[\frac{\underline{Z}^T\underline{Z}+s^2}{C_0}\right]^{\nu-1} \exp\left\{ -\left[\frac{\underline{Z}^T\underline{Z}+s^2}{C_0}\right]^{\nu} \right\} ds \right\}^2. \end{aligned}$$

Now, it suffices to show that

$$0 < E\left\{ \left[\frac{\underline{c}^T\underline{Z}}{f_{\underline{Z}}(\underline{Z})}\right]^2 \left[\int_0^{\infty} \left(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0}\right)^{\nu-1} \exp\left\{ -\left(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0}\right)^{\nu} \right\} ds \right]^2 \right\} < \infty. \quad \text{(A.1)}$$

(I) We first show the above expected value is finite. This is done by considering the following two cases :

Case 1 : If $\nu \le 1$, then

$$E\left\{ \left[\frac{\underline{c}^T\underline{Z}}{f_{\underline{Z}}(\underline{Z})}\right]^2 \left[\int_0^\infty \left(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0}\right)^{\nu-1} \exp\left\{-\left(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0}\right)^\nu\right\} ds\right]^2 \right\}$$

$$\le \quad E\left\{ \left[\frac{\underline{c}^T\underline{Z}}{f_{\underline{Z}}(\underline{Z})}\right]^2 \left[\frac{\underline{Z}^T\underline{Z}}{C_0}\right]^{2\nu-2} \left[\int_0^\infty \exp\left\{-\left(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0}\right)^\nu\right\} ds\right]^2 \right\}$$

$$= \quad \left[\frac{1}{2K_p}\right]^2 \cdot E\left\{ \left[\frac{\underline{c}^T\underline{Z}}{f_{\underline{Z}}(\underline{Z})}\right]^2 \left[\frac{\underline{Z}^T\underline{Z}}{C_0}\right]^{2\nu-2} [f_{\underline{Z}}(\underline{Z})]^2 \right\}$$

$$= \quad \left[\frac{1}{2K_p}\right]^2 \cdot E\left\{ (\underline{c}^T\underline{Z})^2 \left(\frac{\underline{Z}^T\underline{Z}}{C_0}\right)^{2\nu-2} \right\}$$

$$= \quad \left[\frac{1}{2K_p}\right]^2 \cdot C_0^{2-2\nu} E\left\{\underline{c}^T\underline{U}\,\underline{U}^T\underline{c}\right\} E\left\{R^{4\nu-2}\right\}$$

(This step follows from the fact that $\underline{Z} = R\underline{U}$, $\underline{Z}^T\underline{Z} = R^2$, and the

independence of R and $\underline{U}$.)

$$= \quad \left[\frac{1}{2K_p}\right]^2 \cdot C_0^{2-2\nu} E\left\{\frac{1}{p-1}\underline{c}^T\underline{c}\right\} E\left\{R^{4\nu-2}\right\}.$$

So now it suffices to show that $E[R^{4\nu-2}] < \infty$. Note that R^2 has density function of the form

$$f_{R^2}(r) = \frac{K_p \pi^{(p-1)/2}}{\Gamma[(p-1)/2]} r^{(p-1)/2-1} \int_{-\infty}^{\infty} \exp\{-[(r+s^2)/C_0]^\nu\} ds.$$

Thus

$$E\left[R^{4\nu-2}\right] = E\left[(R^2)^{2\nu-1}\right]$$

$$= \frac{2K_p \pi^{(p-1)/2}}{\Gamma[(p-1)/2]} \int_0^\infty\int_0^\infty r^{2\nu-1+(p-1)/2-1} \cdot \exp\{-[(r+s^2)/C_0]^\nu\} ds\, dr$$

Letting $r = t^2$, we have

$$E\left\{R^{4\nu-2}\right\} = \frac{4K_p\pi^{(p-1)/2}}{\Gamma[(p-1)/2]}\int_0^\infty\int_0^\infty t^{4\nu+p-2}\exp\{-[(t^2+s^2)/C_0]^\nu\}ds\,dt.$$

Letting

$$s = \sqrt{C_0}\,x^{1/\nu}\sin(\theta) \text{ and } t = \sqrt{C_0}\,x^{1/\nu}\cos(\theta),$$

we have $[(s^2+t^2)/C_0]^\nu = x^2$ and the jacobian of the transformation is $\frac{1}{\nu}C_0\,x^{(2-\nu)/\nu}$.

Thus, the double integral is equal to

$$\int_0^\infty\int_0^{\pi/2}[C_0^{1/2}x^{1/\nu}\cos(\theta)]^{4\nu-4+p}\ \{(1/\nu)C_0x^{(2-\nu)/\nu}\exp(-x^2)\}\,d\theta\,dx$$

$$= \quad (C_0^{2\nu+p/2-1}/\nu)\left\{\int_0^{\pi/2}(\cos(\theta))^{4\nu+p-4}d\theta\right\}\left\{\int_0^\infty x^{(3\nu+p-2)/\nu}\exp(-x^2)dx\right\}.$$

Note that the first integral in the above expression

$$\int_0^{\pi/2}(\cos(\theta))^{4\nu+p-4}\,d\theta < \infty,$$

if $4\nu+p-4 > -1$, i.e., if $4\nu+p > 3$. Also, letting $y = x^2$, we have the second integral

$$\int_0^\infty x^{(3\nu+p-2)/\nu}\exp(-x^2)\,dx = \frac{1}{2}\int_0^\infty y^{(4\nu-2+p)/2\nu-1}\exp(-y)\,dy < \infty,$$

if $(4\nu-2+p)/2\nu > 0$. Note that $4\nu-2+p$ is always positive since we consider $\nu > 0$ and $p \geq 2$. So we have proved that, if $0 < \nu \leq 1$ and $4\nu+p > 3$,

$$E\left\{R^{4\nu-2}\right\} < \infty, \qquad \text{(A.2)}$$

and hence $I_c(f_Z) < \infty$.

That is

$$I_{\underline{c}}(f_{\underline{Z}}) < \infty \begin{cases} \text{if } p \geq 3 \text{ and } 0 < \nu \leq 1 \\ \text{if } p = 2 \text{ and } \frac{1}{4} < \nu \leq 1 \end{cases}. \tag{A.3}$$

Case 2 : If $\nu > 1$,

then we can write the integral in (A.1), provided it exists, as

$$\int_0^\infty \left(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0}\right)^{\nu-1} \exp\left\{-\left(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0}\right)^{\nu}\right\} ds$$

$$= \int_0^1 \left(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0}\right)^{\nu-1} \exp\left\{-\left(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0}\right)^{\nu}\right\} ds + \int_1^\infty \left(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0}\right)^{\nu-1} \exp\left\{-\left(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0}\right)^{\nu}\right\} ds$$

$$= A + B,$$

where

$$\begin{aligned} A &= \int_0^1 \left(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0}\right)^{\nu-1} \exp\left\{-\left(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0}\right)^{\nu}\right\} ds \\ &\leq \int_0^1 \left(\frac{\underline{Z}^T\underline{Z}+1}{C_0}\right)^{\nu-1} \exp\left\{-\left(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0}\right)^{\nu}\right\} ds \text{ (since } \nu > 1) \\ &= \left(\frac{\underline{Z}^T\underline{Z}+1}{C_0}\right)^{\nu-1} \int_0^1 \exp\left\{-\left(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0}\right)^{\nu}\right\} ds, \end{aligned}$$

and

$$\begin{aligned} B &= \int_1^\infty \left(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0}\right)^{\nu-1} \exp\left\{-\left(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0}\right)^{\nu}\right\} ds \\ &\leq \int_1^\infty s\left(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0}\right)^{\nu-1} \exp\left\{-\left(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0}\right)^{\nu}\right\} ds \text{ (since } s \geq 1) \end{aligned}$$

$$= \frac{C_0}{2}\int_1^{\infty}\left(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0}\right)^{\nu-1}\left(\frac{2s}{C_0}\right)\exp\{-(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0})^{\nu}\}\,ds$$

$$= \frac{C_0}{2\nu}[-\exp\{-(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0})^{\nu}\}|_1^{\infty}]$$

$$= \frac{C_0}{2\nu}\exp\{-(\frac{\underline{Z}^T\underline{Z}+1}{C_0})^{\nu}\}.$$

Note that

$$\begin{aligned} f_{\underline{Z}}(\underline{Z}) &= 2K_p\int_0^{\infty}\exp\{-[\frac{\underline{z}^T\underline{z}+s^2}{C_0}]^{\nu}\}ds \\ &\geq 2K_p\int_0^{1}\exp\{-[\frac{\underline{z}^T\underline{z}+s^2}{C_0}]^{\nu}\}ds \\ &\geq 2K_p\int_0^{1}\exp\{-[\frac{\underline{z}^T\underline{z}+1}{C_0}]^{\nu}\}ds \\ &= 2K_p\exp\{-[\frac{\underline{z}^T\underline{z}+1}{C_0}]^{\nu}\}. \end{aligned}$$

Therefore

$$\frac{A}{f_{\underline{Z}}(\underline{Z})} \leq [\frac{\underline{z}^T\underline{z}+1}{C_0}]^{\nu-1}\cdot\frac{1}{2K_p},\ \text{and}\ \frac{B}{f_{\underline{Z}}(\underline{Z})} \leq \frac{C_0}{2\nu}\,\frac{1}{2K_p}.$$

Thus, by (A.1), we have

$$E\left\{[\frac{\underline{c}^T\underline{Z}}{f_{\underline{Z}}(\underline{Z})}]^2[A+B]^2\right\}$$

$$\leq \frac{1}{4K_p^2}E\left\{[(\underline{c}^T\underline{Z})^2[(\frac{\underline{z}^T\underline{z}+1}{C_0})^{\nu-1}+\frac{C_0}{2\nu}]^2\right\}$$

$$= \frac{1}{4K_p^2}E\left\{\underline{c}^T\underline{U}\,\underline{U}^T\underline{c}\right\}E\left\{R^2[(\frac{R^2+1}{C_0})^{\nu-1}+\frac{C_0}{2\nu}]^2\right\}$$

(Since $\underline{Z} = R\underline{U}$, $\underline{Z}^T\underline{Z} = R^2$, and R is independent of and $\underline{U}$.)

So now it suffice to show that

$$E\left\{ R^2[(\frac{R^2+1}{C_0})^{\nu-1} + \frac{C_0}{2\nu}]^2 \right\} < \infty.$$

Let

$$g(R^2) = R^2[(\frac{R^2+1}{C_0})^{\nu-1} + \frac{C_0}{2\nu}]^2 = \{R[(\frac{R^2+1}{C_0})^{\nu-1} + \frac{C_0}{2\nu}]\}^2.$$

Note that g is a function of R^2 of order 2ν -1. Using the result that

" If $X \in \mathcal{L}_p$, then $X \in \mathcal{L}_q$ for all $0 < q < p$.",

it suffice to show that $E\left[(R^2)^{2\nu-1}\right] < \infty$. From (A.2), we see that the result holds if $4\nu+p > 3$. Note that $4\nu+p > 3$ is always true since we are now considering $\nu > 1$ and $p \geq 2$. Thus,

$$I_{\underline{c}}(f_{\underline{Z}}) < \infty \quad \text{if } \nu > 1. \tag{A.4}$$

Combining (A.3) and (A.4), we have proved $I_{\underline{c}}(f_{\underline{Z}}) < \infty$ if $4\nu+p > 3$. That is

$$I_{\underline{c}}(f_{\underline{Z}}) < \infty \begin{cases} \text{if } p \geq 3 \text{ and } \nu > 0 \\ \text{if } p = 2 \text{ and } \nu > \frac{1}{4} \end{cases}. \tag{A.5}$$

(II) We now need to show $I_{\underline{c}}(f_{\underline{Z}}) > 0$.

From (A.1), it suffices to show that

$$E\left\{ [\frac{\underline{c}^T\underline{Z}}{f_{\underline{Z}}(\underline{Z})}]^2 [\int_0^\infty (\frac{\underline{Z}^T\underline{Z}+s^2}{C_0})^{\nu-1} \exp\{-(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0})^\nu\} \, ds]^2 \right\} > 0.$$

Since $f_{\underline{Z}}(\underline{Z}) > 0$ and $h(\underline{Z}) = \int_0^\infty (\frac{\underline{Z}^T\underline{Z}+s^2}{C_0})^{\nu-1} \exp\{-(\frac{\underline{Z}^T\underline{Z}+s^2}{C_0})^{\nu}\}\, ds > 0\ \forall\ \underline{Z}$, we need to show

$$E\{(\underline{c}^T\underline{Z})\, g(\underline{Z})\}^2 > 0\ \forall\ \underline{c} \neq \underline{0}, \text{ where } g(\underline{Z}) = h(\underline{Z})/f_{\underline{Z}}(\underline{Z}). \tag{A.6}$$

To verify (A.6), we note that if $\exists\ \underline{c} \neq \underline{0}$ s.t. $E\{(\underline{c}^T\underline{Z})\, g(\underline{Z})\}^2 \leq 0$

$\Rightarrow \{(\underline{c}^T\underline{Z})\, g(\underline{Z})\}^2 = 0$ a.e.

$\Rightarrow (\underline{c}^T\underline{Z})\, g(\underline{Z}) = 0$ a.e.

$\Rightarrow (\underline{c}^T\underline{Z}) = 0$ a.e. (since $g(\underline{Z}) > 0\ \forall\ \underline{Z}$)

$\Rightarrow (\underline{c}^T\underline{U}) = 0$ a.e. (since $\underline{Z} = R\underline{U}$ and $R > 0$)

$\Rightarrow \mathrm{Var}(\underline{c}^T\underline{U}) = 0$

$\Rightarrow \sum_{j=1}^{p-1} c_j^2\, \mathrm{Var}(U_j) = 0$

(Since $\underline{c}^T = (c_1, \ldots, c_{p-1})$, $\underline{U}^T = (U_1, \ldots, U_{p-1})$, and U_j's are uncorrelated.)

$\Rightarrow c_j = 0\ \forall\ j = 1, \ldots, p-1$. A contradiction.

This completes the proof.

APPENDIX B

Here, we prove in the context of Theorem 5.4.4 that if any one of the pairs (i, k), (i', k'), (s, t) or (s', t') is disjoint from the integers in the others, then the expected value

$$E_{H_0}\left\{[\cos(\pi\,\hat{p}_{ik,i'k'}) - \cos(\alpha_{ik,i'k'})]\cdot[\cos(\pi\,\hat{p}_{st,s't'}) - \cos(\alpha_{st,s't'})]\right\}$$

is zero.

<u>Proof</u> Suppose (i, k) is unique (i.e. disjoint), then

$$E\left\{[\cos(\pi\hat{p}_{ik,i'k'})-\cos(\alpha_{ik,i'k'})]\cdot[\cos(\pi\hat{p}_{st,s't'})-\cos(\alpha_{st,s't'})]\right\}$$

$$= E\Big\{E_{\underline{X}_i,\,\underline{X}_k}\Big([\cos(\pi\hat{p}_{ik,i'k'})-\cos(\alpha_{ik,i'k'})]\cdot$$

$$[\cos(\pi\hat{p}_{st,s't'})-\cos(\alpha_{st,s't'})] \mid \text{all } \underline{X}_j \text{ except } j = i \text{ and } k\Big)\Big\}. \tag{B.1}$$

Note that

$$E_{\underline{X}_i,\,\underline{X}_k}\left\{[\cos(\pi\hat{p}_{ik,i'k'})-\cos(\alpha_{ik,i'k'})]\cdot\right.$$

$$\left.[\cos(\pi\hat{p}_{st,s't'})-\cos(\alpha_{st,s't'})] \mid \text{all } \underline{X}_j \text{ except } j = i \text{ and } k\right\}$$

$$= E_{D_i,\,\underline{A}_i,\,D_k,\,\underline{A}_k}\Big\{\big[\cos(\pi\hat{p}(D_i\underline{A}_i, D_k\underline{A}_k, \underline{X}_{i'}, \underline{X}_{k'}))-\cos(\alpha(D_i\underline{A}_i, D_k\underline{A}_k, \underline{X}_{i'},$$

$$\underline{X}_{k'}))\big]\cdot\big[\cos(\pi\hat{p}_{st,s't'})-\cos(\alpha_{st,s't'})\big] \mid \text{all } \underline{X}_j \text{ except } j = i \text{ and } k\Big\}$$

$$= E_{\underline{A}_i,\,\underline{A}_k}\Big\{E_{D_i,\,D_k}\big[\cos(\pi\hat{p}(D_i\underline{A}_i, D_k\underline{A}_k, \underline{X}_{i'}, \underline{X}_{k'}))-\cos(\alpha(D_i\underline{A}_i, D_k\underline{A}_k, \underline{X}_{i'},$$

$$\underline{X}_{k'})) \mid \underline{A}_i, \underline{A}_k\big]\cdot\big[\cos(\pi\hat{p}_{st,s't'})-\cos(\alpha_{st,s't'})\big] \mid \text{all } \underline{X}_j \text{ except } j = i \text{ and } k\Big\}, \tag{B.2}$$

where $D_i = \text{sign}(X_{i1})$, $\underline{A}_i = \text{sign}(X_{i1})\underline{X}_i$. Now conditioning on $\underline{A}_i$, $\underline{A}_k$ and all $\underline{X}_j$ with $j \neq i$ or k, we have

$$E_{D_i, D_k}\left[\cos(\alpha(D_i\underline{A}_i, D_k\underline{A}_k, \underline{X}_{i'}, \underline{X}_{k'}))\right]$$

$$\begin{aligned} = \quad & \left\{\cos(\alpha(\underline{A}_i, \underline{A}_k, \underline{X}_{i'}, \underline{X}_{k'}))\cdot P(D_i = 1, D_k = 1)\right. \\ & + \cos(\alpha(\underline{A}_i, -\underline{A}_k, \underline{X}_{i'}, \underline{X}_{k'}))\cdot P(D_i = 1, D_k = -1) \\ & + \cos(\alpha(-\underline{A}_i, \underline{A}_k, \underline{X}_{i'}, \underline{X}_{k'}))\cdot P(D_i = -1, D_k = 1) \\ & \left. + \cos(\alpha(-\underline{A}_i, -\underline{A}_k, \underline{X}_{i'}, \underline{X}_{k'}))\cdot P(D_i = -1, D_k = -1)\right\}. \end{aligned} \tag{B.3}$$

Note that D_i and D_k are conditionally i.i.d. with $P(D_i = 1) = P(D_i = -1) = \frac{1}{2}$. Note also that

$$\cos(\alpha(\underline{A}_i, \underline{A}_k, \underline{X}_{i'}, \underline{X}_{k'})) = -\cos(\alpha(-\underline{A}_i, \underline{A}_k, \underline{X}_{i'}, \underline{X}_{k'})),$$

$$\text{and} \quad \cos(\alpha(-\underline{A}_i, -\underline{A}_k, \underline{X}_{i'}, \underline{X}_{k'})) = -\cos(\alpha(\underline{A}_i, -\underline{A}_k, \underline{X}_{i'}, \underline{X}_{k'})).$$

Thus, the expression in (B.3) equals 0. Similarly,

$$\cos(\pi\hat{p}(\underline{A}_i, \underline{A}_k, \underline{X}_{i'}, \underline{X}_{k'})) = -\cos(\pi\hat{p}(-\underline{A}_i, \underline{A}_k, \underline{X}_{i'}, \underline{X}_{k'})),$$

$$\text{and} \quad \cos(\pi\hat{p}(-\underline{A}_i, -\underline{A}_k, \underline{X}_{i'}, \underline{X}_{k'})) = -\cos(\pi\hat{p}(\underline{A}_i, -\underline{A}_k, \underline{X}_{i'}, \underline{X}_{k'})),$$

since, for example, $\hat{p}(-\underline{A}_i, \underline{A}_k, \underline{X}_{i'}, \underline{X}_{k'}) = 1-\hat{p}(\underline{A}_i, \underline{A}_k, \underline{X}_{i'}, \underline{X}_{k'})$. Therefore, following from the same arguments made above, conditioning on $\underline{A}_i$, $\underline{A}_k$ and all $\underline{X}_j$ with $j \neq i$ or k,

$$E_{D_i, D_k}\left[\cos(\pi\hat{p}(D_i\underline{A}_i, D_k\underline{A}_k, \underline{X}_{i'}, \underline{X}_{k'}))\right] = 0. \tag{B.4}$$

Hence, using expressions (B.2) - (B.4), expression (B.1) implies that

$$E_{H_0}\left\{\left[\cos(\pi\,\hat{p}_{ik,i'k'}) - \cos(\alpha_{ik,i'k'})\right]\cdot\left[\cos(\pi\,\hat{p}_{st,s't'}) - \cos(\alpha_{st,s't'})\right]\right\} = 0.$$

This completes the proof.

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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